Categorification of the singular braid monoids and of the virtual braid groups

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Summary

1. Soergel bimodules
   - Definition
   - Two morphisms
   - Tensoring Soergel bimodules
2. Categorification of the $\mathcal{B}_n$ and its generalization to $\mathcal{SB}_n$
   - Categorification of the braid groups
   - Categorification of the singular braid monoids
3. Categorification of $\mathcal{VB}_n$

Overview

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   - Definition
   - Two morphisms
   - Tensoring Soergel bimodules
2. Categorification of the $\mathcal{B}_n$ and its generalization to $\mathcal{SB}_n$
   - Categorification of the braid groups
   - Categorification of the singular braid monoids
3. Categorification of $\mathcal{VB}_n$

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Let R be the subalgebra of $\mathbb{Q}[x_1, \ldots, x_n]$ defined by

$$R = \mathbb{Q}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n].$$

The action of S_n preserves R. Let R^H be the subalgebra of elements of R fixed by a subgroup H of S_n. In particular R^{τ_i} is the subalgebra of R of elements fixed by the transposition $\tau_i = (i, i+1)$.

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We introduce a grading on R, R^{τ_i} and B_i by setting

$$\deg(x_k) = 2.$$

If $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a $\mathbb{Z}$–graded bimodule and p an integer then the shifted bimodule $M\{p\}$ is defined by $M\{p\}_i = M_{i-p}$.

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Definition

Soergel bimodules are direct summands of shifted tensor products of B_i's.

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$$\begin{array}{cccc} \mathrm{br}_i : & B_i & \longrightarrow & R \\ & 1 \otimes 1 & \longmapsto & 1 \end{array}$$

$$\begin{array}{cccc} \mathrm{rb}_i : & R\{2\} & \longrightarrow & B_i \\ & 1 & \longmapsto & X_i \otimes 1 + 1 \otimes X_i \end{array}$$

Since $R \cong R^{\tau_i} \oplus R^{\tau_i}\{2\}$ as graded R^{τ_i}–modules, the morphism rb_i is well-defined (*ie* $p\,\mathrm{rb}_i(1) = \mathrm{rb}_i(1)p$ for all $p \in R$).

$$p\,\mathrm{rb}_i(1) \quad = \quad (a + bX_i)(X_i \otimes 1 + 1 \otimes X_i)$$

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&= (X_i\otimes 1+1\otimes X_i)(a+bX_i)\\
&= \mathrm{rb}_i(1)p.
\end{aligned}$$

Three isomorphisms

Theorem (Soergel)

There are isomorphims of graded R–bimodules:

$$\begin{aligned}
B_i \otimes_R B_i &\cong B_i \oplus B_i\{2\}, \\
B_i \otimes_R B_j &\cong B_j \otimes_R B_i \text{ for } |i-j| > 1, \\
B_i \otimes_R B_{i+1} \otimes_R B_i &\cong B_{i,i+1} \oplus B_i\{2\}, \\
B_{i+1} \otimes_R B_i \otimes_R B_{i+1} &\cong B_{i,i+1} \oplus B_{i+1}\{2\} \text{ so} \\
B_i \otimes_R B_{i+1} \otimes_R B_i \oplus B_{i+1}\{2\} &\cong B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \oplus B_i\{2\}
\end{aligned}$$

where $B_{i,i+1} = R \otimes_{R^{<\tau_i,\tau_{i+1}>}} R$.

$B_i \otimes_R B_i \cong B_i \oplus B_i\{2\}$

The bimodule B_i injects in two different ways into $B_i \otimes B_i$; either

$$1 \otimes 1 \longmapsto 1 \otimes 1 \otimes 1$$

or

$$1 \otimes 1 \longmapsto 1 \otimes X_i \otimes 1.$$

The two elements $1 \otimes 1 \otimes 1$ and $1 \otimes X_i \otimes 1$ span $B_i \otimes B_i$ as a R–bimodule.

$B_i \otimes_R B_j \cong B_j \otimes_R B_i$

If $|i-j|>1$, the bimodule $B_i \otimes_R B_j$ is spanned by $1 \otimes 1 \otimes 1$ as a R–bimodule, so the isomorphism between $B_i \otimes_R B_j$ and $B_j \otimes_R B_i$ is entirely defined by the image of $1 \otimes 1 \otimes 1$:

$$1 \otimes 1 \otimes 1 \longmapsto 1 \otimes 1 \otimes 1$$

$B_i \otimes_R B_{i+1} \otimes_R B_i \cong B_{i,i+1} \oplus B_i\{2\}$

The bimodule B_i injects into $B_i \otimes_R B_{i+1} \otimes_R B_i$ in the following way:

$$\begin{array}{ccccc} B_i\{2\} & \longrightarrow & B_i \otimes_R B_i\{2\} & \longrightarrow & B_i \otimes_R B_{i+1} \otimes_R B_i \\ 1 \otimes 1 & \longmapsto & 1 \otimes 1 \otimes 1 & \longmapsto & 1 \otimes X_{i+1} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_{i+1} \otimes 1 \end{array}$$

Since $R^{<\tau_i,\tau_{i+1}>} \simeq R^{\tau_i} \cap R^{\tau_{i+1}}$, the following injection is well-defined:

$$\begin{array}{ccc} B_{i,i+1} & \longrightarrow & B_i \otimes_R B_{i+1} \otimes_R B_i \\ 1 \otimes 1 & \longmapsto & 1 \otimes 1 \otimes 1 \otimes 1 \end{array}$$

The bimodule $B_i \otimes_R B_{i+1} \otimes_R B_i$ is the direct sum of the images of these two injections.

Overview

Braid groups

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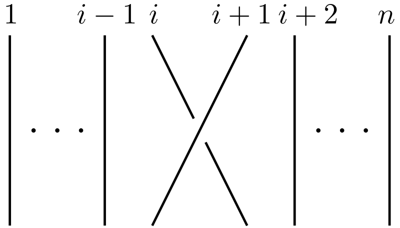

Figure: The positive elementary braid σ_i

Braid groups

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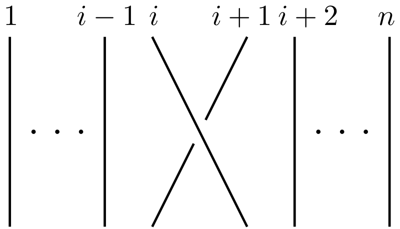

Figure: The negative elementary braid σ_i^{-1}

Braid groups

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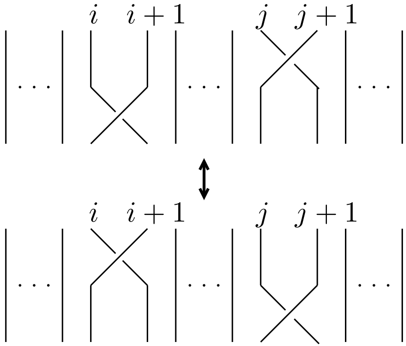
$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1,$$



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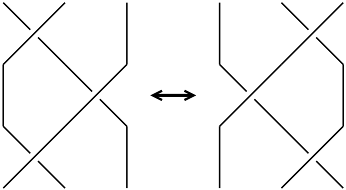

Categorification of $\mathcal{B}_n$

To each braid generator $\sigma_i \in \mathcal{B}_n$ we assign the cochain complex $F(\sigma_i)$ of graded R-bimodules

$$F(\sigma_i) : 0 \longrightarrow \underset{-1}{R\{2\}} \xrightarrow{\mathrm{rb}_i} \underset{0}{B_i} \longrightarrow 0$$

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To the unit element 1 we assign the complex of graded R-bimodules

$$F(1) : 0 \longrightarrow \underset{0}{R} \longrightarrow 0,$$

Categorification of $\mathcal{B}_n$

To any word $\sigma = \sigma_{i_1}^{\varepsilon_1} \ldots \sigma_{i_k}^{\varepsilon_k}$ where $\varepsilon_1, \ldots, \varepsilon_k = \pm 1$, we assign the complex of graded R-bimodules

$$F(\sigma) = F(\sigma_{i_1}^{\varepsilon_1}) \otimes_R \cdots \otimes_R F(\sigma_{i_k}^{\varepsilon_k}).$$

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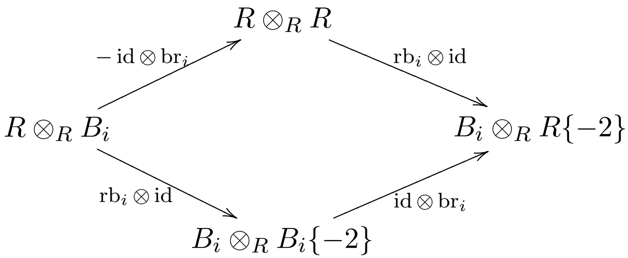
Example given

$F(\sigma_i \sigma_i^{-1})$:



Categorification of $\mathcal{B}_n$

Rouquier proved the following result, which is called a categorification of the braid group $\mathcal{B}_n$.

Theorem (Rouquier)

If ω and ω' are words representing the same element of $\mathcal{B}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of graded R-bimodules.

$$F(\sigma_i\sigma_i^{-1}) : \quad R\otimes_R B_i \xrightarrow{\begin{pmatrix} -\mathrm{id}\otimes \mathrm{br}_i \\ \mathrm{rb}_i\otimes \mathrm{id}\end{pmatrix}} R\otimes_R R \oplus B_i\otimes_R B_i\{-2\} \xrightarrow{\begin{pmatrix} \mathrm{rb}_i\otimes \mathrm{id} & \mathrm{id}\otimes \mathrm{br}_i\end{pmatrix}} B_i\otimes_R R\{-2\}$$

$$F(1) : \quad 0 \longrightarrow R \longrightarrow 0$$

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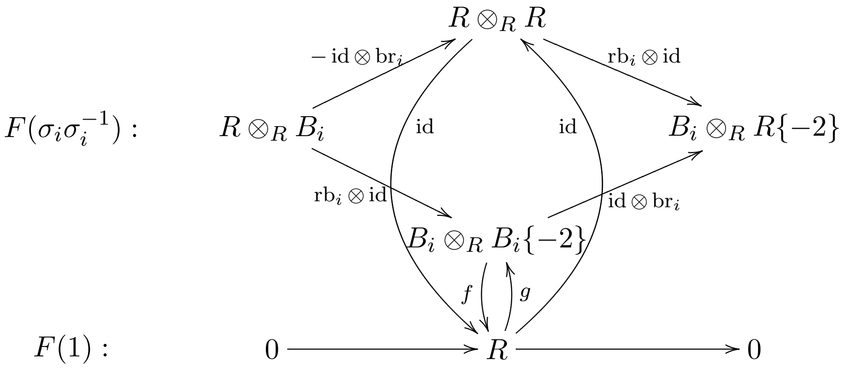
$$F(1): \quad 0 \longrightarrow R \longrightarrow 0$$

$g\circ f-\mathrm{id}=d\circ h+h\circ d$ and $f\circ g-\mathrm{id}=d\circ h+h\circ d$



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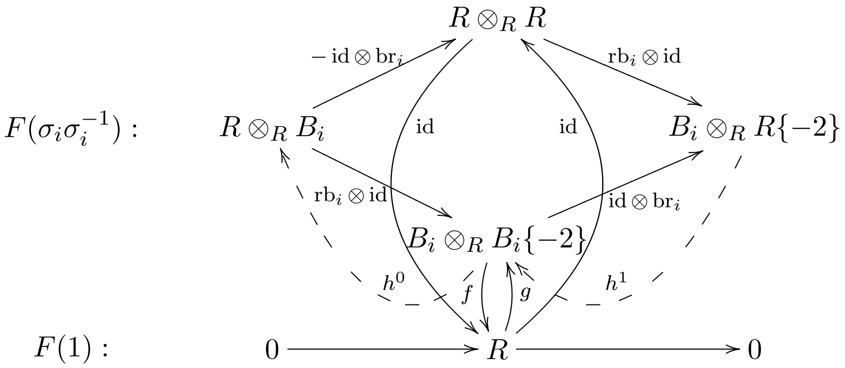

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Singular braid monoids

The singular braid monoid $\mathcal{SB}_n$ is the monoid generated by $3(n-1)$ generators σ_i, σ_i^{-1} and ρ_i, for $i = 1, \ldots, n-1$ which can be diagrammatically depicted by

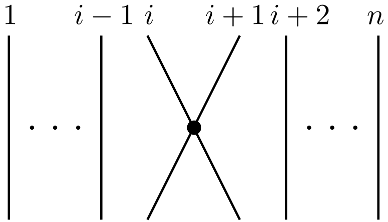

Figure: The singular elementary braid ρ_i

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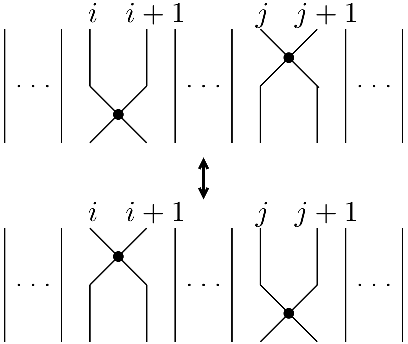

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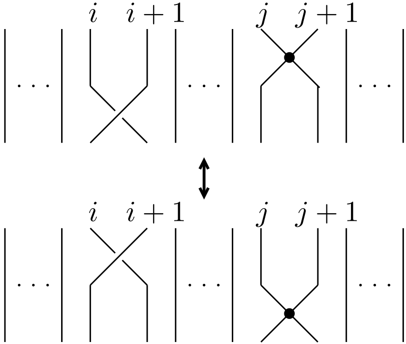

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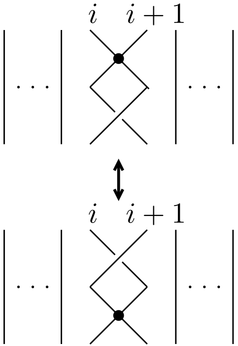

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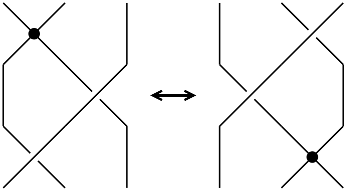

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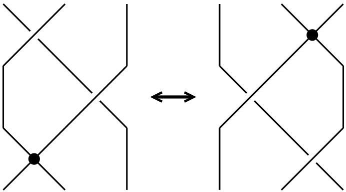

Categorification of $\mathcal{SB}_n$

To the generators σ_i and σ_i^{-1} of $\mathcal{SB}_n$ coming from $\mathcal{B}_n$ we assign Rouquier's complexes $F(\sigma_i)$ and $F(\sigma_i^{-1})$.
To the generator ρ_i we assign the cochain complex $F(\rho_i)$ of graded R–bimodules

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Theorem

If ω and ω' are words representing the same element of $\mathcal{SB}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of R–bimodules.

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Virtual braid groups

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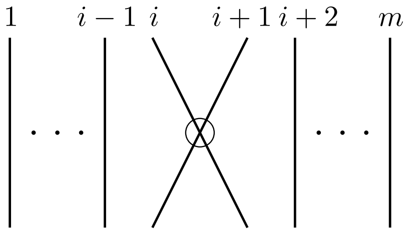

Figure: The virtual elementary braid ζ_i

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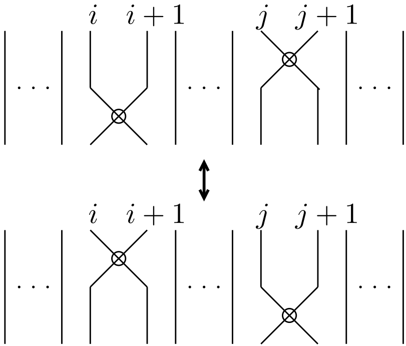

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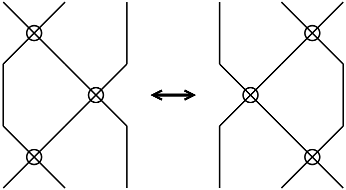

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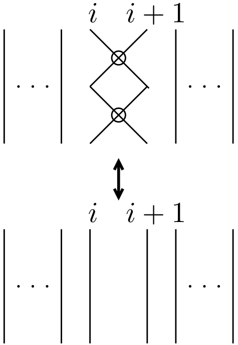

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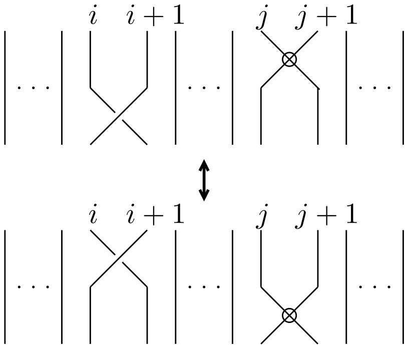

Virtual braid groups

The virtual braid group $\mathcal{VB}_n$ is the group generated by $2(n-1)$ generators σ_i and ζ_i for $i = 1, \ldots, n-1$ such that the generators σ_i satisfy the braid relations and the following relations are verified

$$\zeta_i \zeta_j = \zeta_j \zeta_i \text{ for } |i-j| > 1,$$

$$\zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1},$$

$$\zeta_i^2 = 1,$$

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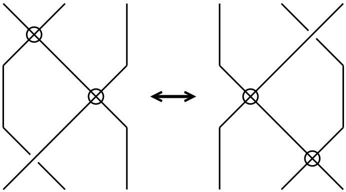

Twisted bimodules

For each permutation ω in S_n we consider the R-bimodule R_ω: as a left R-module, R_ω is equal to R

$$a.p = ap \text{ for all } p \in R_\omega, \; a \in R$$

while the right action of $a \in R$ is the multiplication by $\omega(a)$

$$p.a = p\omega(a) \text{ for all } p \in R_\omega, \; a \in R.$$

Twisted bimodules

Lemma

For all $\omega, \omega' \in S_n$ there is an isomorphism of R-bimodules

$$\begin{array}{ccc} R_\omega \otimes_R R_{\omega'} & \longrightarrow & R_{\omega\omega'} \\ a \otimes b & \longmapsto & a\omega(b) \end{array}$$

Example given $R_{\tau_i} \otimes_R R_{\tau_i} \cong R$.

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Lemma

For all $\omega, \omega' \in S_m$ the R-bimodules $R_\omega \otimes_{R^{\omega'}} R$ and $R \otimes_{R^{\omega\omega'\omega^{-1}}} R_\omega$ are isomorphic.

Example given
$R_{\tau_j} \otimes_R B_i \cong R_{\tau_j} \otimes_{R^{\tau_i}} R \cong R \otimes_{R^{\tau_j \tau_i \tau_j}} R_{\tau_j} \cong R \otimes_{R^{\tau_i}} R_{\tau_j} \cong B_i \otimes_R R_{\tau_j}$
for $|i - j| > 1$.

Categorification of $\mathcal{VB}_n$

To the generators σ_i and σ_i^{-1} of $\mathcal{VB}_n$ coming from $\mathcal{B}_n$ we assign Rouquier's complexes $F(\sigma_i)$ and $F(\sigma_i^{-1})$.
To the generator ζ_i we assign the cochain complex $F(\zeta_i)$ of graded R–bimodules

$$F(\zeta_i) : 0 \longrightarrow \underset{0}{R_{\tau_i}} \longrightarrow 0.$$

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Theorem

If ω and ω' are words representing the same element of $\mathcal{VB}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of R–bimodules.

Thank you.