


## Zariski-Van Kampen Method

Purpose:

Obtain a presentation for the fundamental group of the complement of a plane projective curve in  $\mathbb{P}^2$ .

We will put together several ingredients, among which, the *Van Kampen Theorem* is key.



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**Theorem**

$\pi_1(X, x_0) = \pi_1(F_p, x_0) \rtimes \pi_1(M, p)$ , where the action of  $\pi_1(M, p)$  on  $\pi_1(F_p, x_0)$  is given by the monodromy of  $\pi$ .

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#### Proposition

*Meridians around the same irreducible components of  $B$  are conjugate in  $\pi_1(M \setminus B)$ . Moreover, the conjugacy class of a meridian coincides with the set of homotopy classes of meridians around the same irreducible component.*

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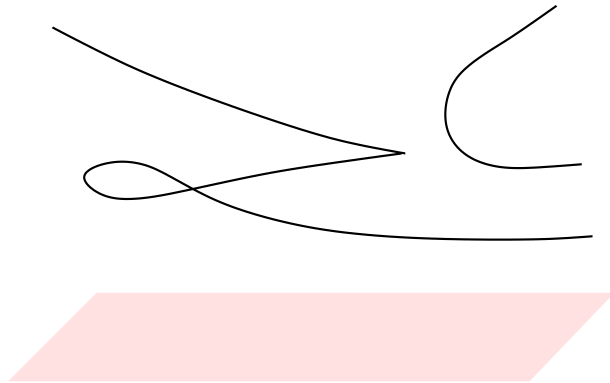
The inclusion  $M \setminus B \hookrightarrow M$  induces a surjective morphism, whose kernel is given by the smallest normal subgroup of  $\pi_1(M \setminus B)$  containing meridians of all the irreducible components of  $B$ .

## Zariski-Van Kampen Theorem

Let  $C \subset \mathbb{P}^2$  be a projective plane curve. Consider  $P = [0 : 1 : 0] \in \mathbb{P}^2 \setminus C$ .

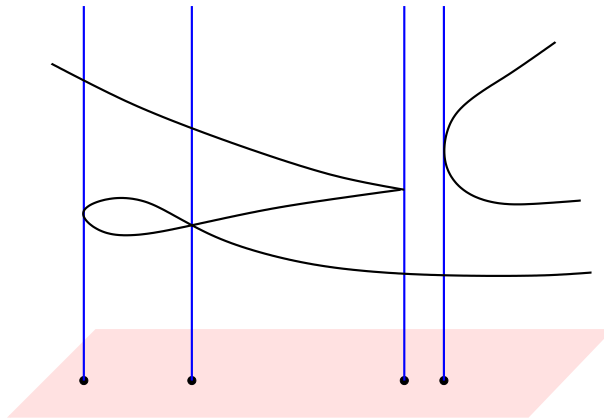
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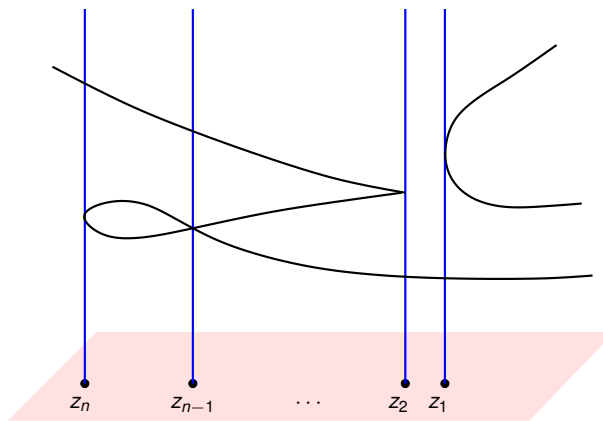
Let  $C \subset \mathbb{P}^2$  be a projective plane curve. Consider  $P = [0 : 1 : 0] \in \mathbb{P}^2 \setminus C$ . Project  $\pi : \mathbb{P}^2 \setminus \{P\} \rightarrow \mathbb{P}^1$  from  $P$





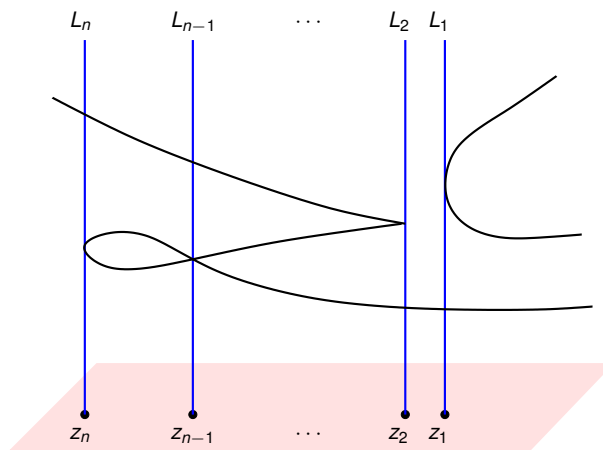
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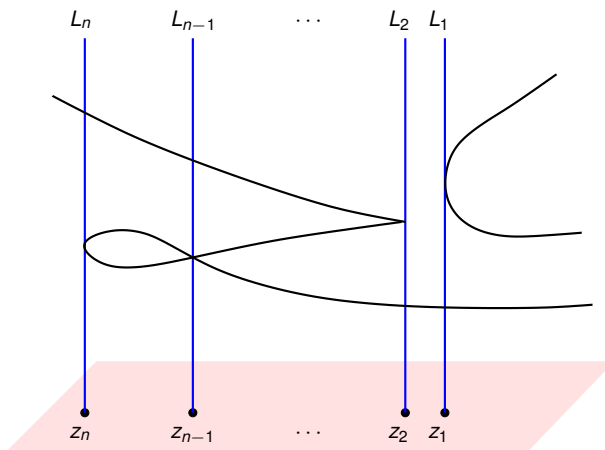


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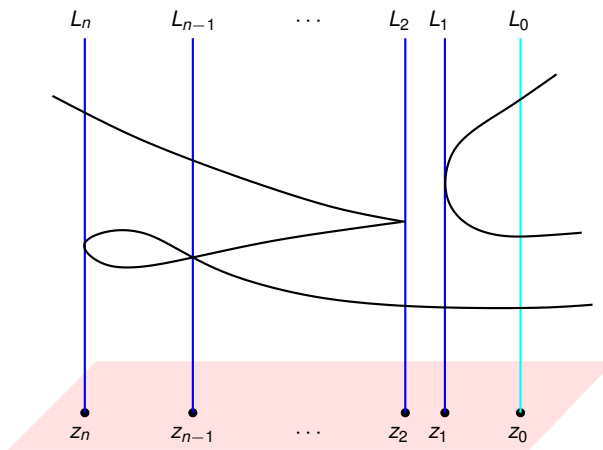
## Zariski-Van Kampen Theorem



### Remark (1)

Let  $X = \mathbb{P}^2 \setminus (\mathcal{C} \cup L)$ , then  $\pi|_X : X \rightarrow \mathbb{P}^1 \setminus Z_n$  is a locally trivial fibration.

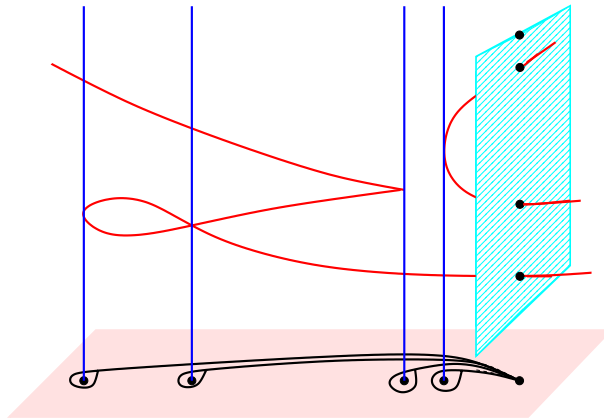
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Let  $X = \mathbb{P}^2 \setminus (\mathcal{C} \cup L)$ , then  $\pi|_X : X \rightarrow \mathbb{P}^1 \setminus Z_n$  is a locally trivial fibration. Moreover, its fiber is  $\mathbb{P}^1 \setminus Z_d$ , where  $d := \deg \mathcal{C}$ .

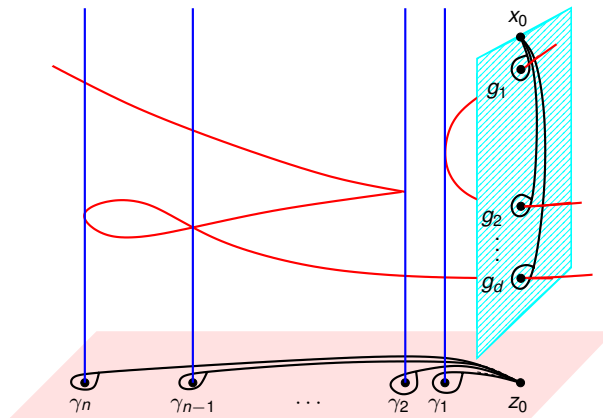
## Zariski-Van Kampen Theorem



### Remark (2)

By (2.1),  $\pi_1(X, x_0) = \pi_1(F_{z_0}, x_0) \rtimes \pi_1(\mathbb{P}^1 \setminus Z_n, z_0)$ . Action is given by the monodromy of  $\pi_1(\mathbb{P}^1 \setminus Z_n, z_0)$  on  $\pi_1(F_{z_0}, x_0)$ .

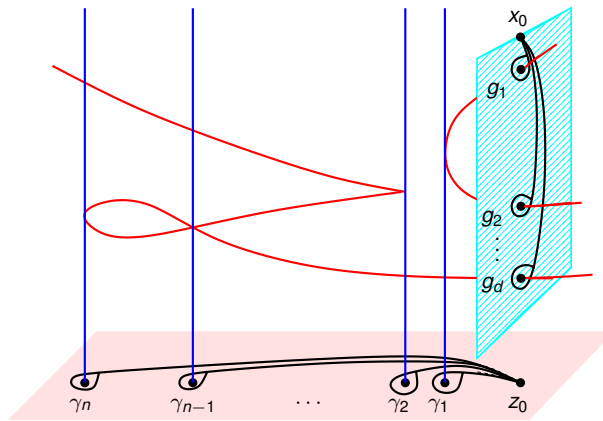
## Zariski-Van Kampen Theorem



### Remark (3)

Note that  $\pi_1(F_{z_0}, x_0) = \langle g_1, \dots, g_d : g_d g_{d-1} \cdots g_1 = 1 \rangle$  and  
 $\pi_1(\mathbb{P}^1 \setminus Z_n, z_0) = \langle \gamma_1, \dots, \gamma_n : \gamma_n \cdots \gamma_1 = 1 \rangle$ .

## Zariski-Van Kampen Theorem

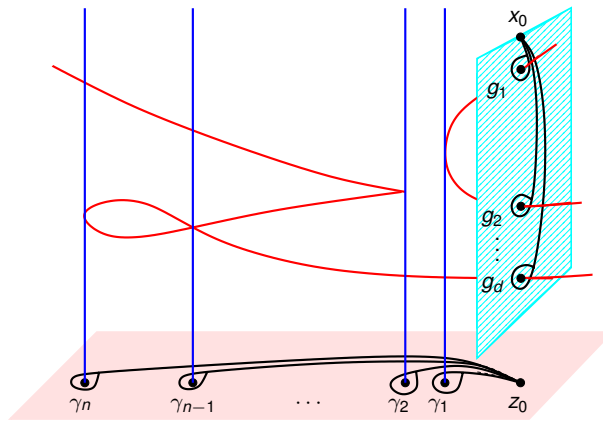


### Theorem

$\pi_1(X, x_0)$  admits the following presentation:

$$\langle g_1, \dots, g_d, \gamma_1, \dots, \gamma_n : g_d g_{d-1} \cdots g_1 = \gamma_n \cdots \gamma_1 = 1, g_i^{\gamma_j} = \gamma_j^{-1} g_i \gamma_j \rangle$$

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### Theorem

$\pi_1(\mathbb{P}^2 \setminus C)$  admits the following presentation:

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Remark

- Let  $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_r$  the decomposition of  $\mathcal{C}$  in its irreducible components, then

$$H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/(d_1, \dots, d_r),$$

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- If two curves are in a connected family of equisingular curves, then they are isotopic

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$\mathcal{C}$  smooth of degree  $d \Rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$ .

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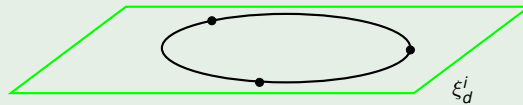
$\mathcal{C}$  smooth of degree  $d \Rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$ . For computation purposes it is more convenient to use a *non-generic* projection. Use for instance  $\mathcal{C} := \{F = 0\}$ , where  $F(X, Y, Z) = X^d + Y^d - Z^d$ .  $P = [0 : 1 : 0] \notin \mathcal{C}$  and  $F_Y = dY^{d-1}$ .

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Let us compute the local monodromy of  $x = y^d$ . Consider  $\gamma(t) = e^{2\pi t\sqrt{-1}}$  a loop around  $x = 0$ . The fiber at  $\gamma(t)$  is given by:

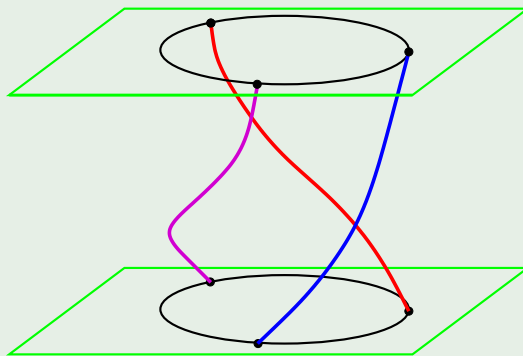


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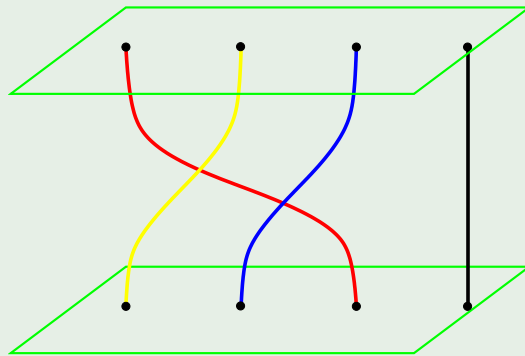
The monodromy around  $x = 0$  looks as follows:



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Corresponds to the braid  $\sigma_1 \sigma_2 \cdots \sigma_{d-1}$



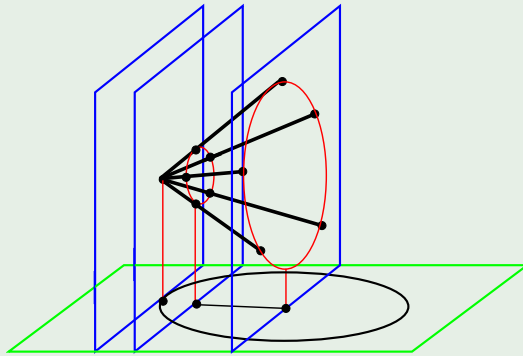


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Note that the global part of the monodromy has no contribution:

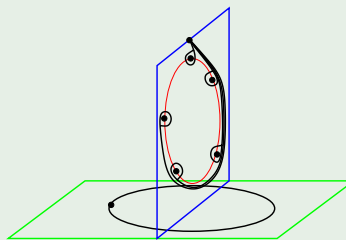


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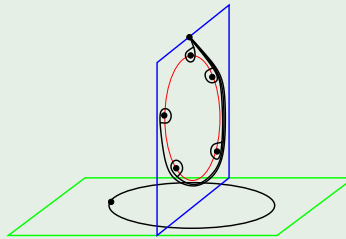


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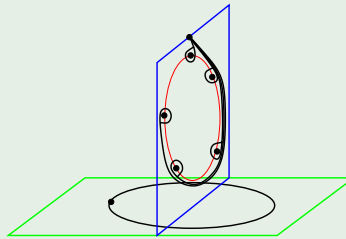
$$g_i = g_i^{(\sigma_1 \sigma_2 \cdots \sigma_{d-1})} = \begin{cases} g_d & i = 1 \\ g_d^{-1} g_{i-1} g_d & i \neq 1 \end{cases}$$

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hence  $g_2 = g_d^{-1} g_1 g_d = g_1$ , and by induction  $g_1 = \dots = g_d = g$ . Finally,  $g_1 \dots g_d = 1$  becomes  $g^d = 1$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \langle g : g^d = 1 \rangle = \mathbb{Z}/d\mathbb{Z}.$$

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Example (Zariski)

Let  $C$  be a general nodal rational curve of degree  $d$ . Consider  $\tilde{C}$  its dual. Note that  $\tilde{C}$  is a rational curve of degree  $2(d-1)$ ,  $2(d-2)(d-3)$  nodes, and  $3(d-2)$  cusps. The fundamental group of  $\tilde{C}$  coincides with the fundamental group of the unordered configuration space of  $d$  points in  $\mathbb{S}^2$ , that is,

$$\mathbb{B}_d(\mathbb{S}^2) = \langle g_1, \dots, g_{d-1} : \begin{array}{l} g_i g_j = g_j g_i, \\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \\ g_1 \cdots g_{d-2} g_{d-1}^2 g_{d-2} \cdots g_1 = 1 \end{array} \rangle.$$

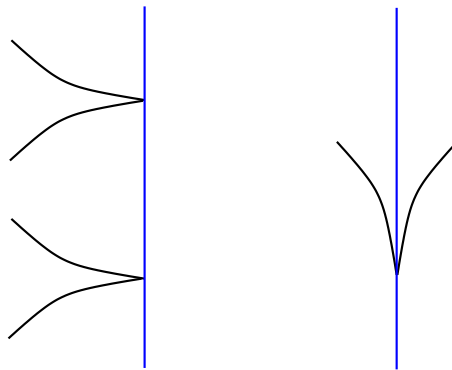


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- Computational methods are effective essentially over  $\mathbb{Z}[\sqrt{-1}]$ .



Example

Consider the following quartic:

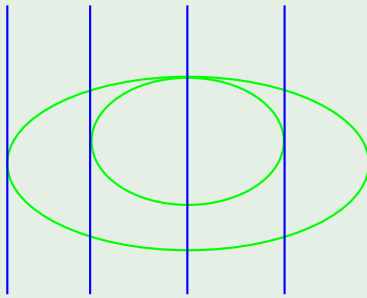
Two concentric green ellipses centered on the page. The inner ellipse is smaller and more circular, while the outer ellipse is larger and more elongated horizontally.





Example

Consider the following quartic and project from  $[0 : 1 : 0]$





Example

Compute the braid monodromy:

The diagram shows a braid with four vertical blue strands. A red rectangle encloses the first two strands. A green loop encircles the second and third strands. A black line with four dots connects the top of the strands to the bottom, with a crossing between the first and second strands.



Example

Compute the braid monodromy:  $\sigma_1^8$ ,

The diagram shows a braid with four vertical blue strands. The second strand from the left is enclosed in a red rectangular box. A green oval encircles the second and third strands. A black line with four dots (representing the strands) starts at the top of the red box and moves downwards, crossing the strands in a sequence that represents the eighth power of the first strand crossing the second strand.



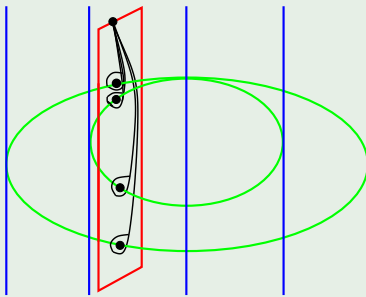
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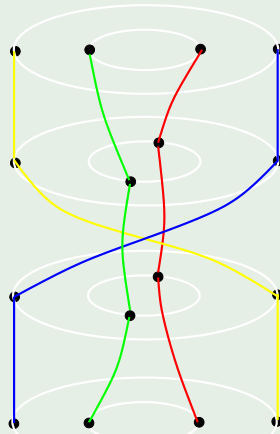
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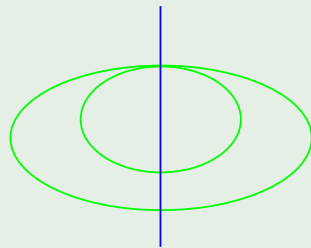
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$$g_1^{\sigma_1^8} = (g_2 g_1)^4 g_1 (g_2 g_1)^{-4} \Rightarrow [(g_2 g_1)^4, g_1] = 1$$



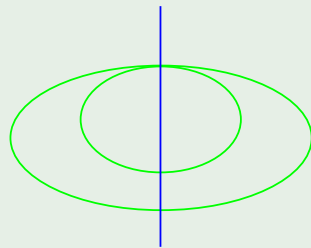


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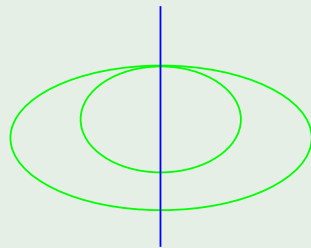
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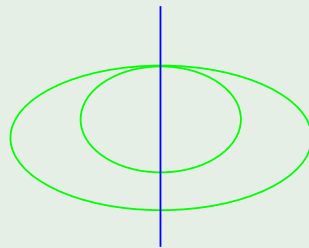
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$$g_4^{\sigma_1^8} = g_4$$





Example

$\sigma_2$ :

A diagram on a light green background. It features two overlapping green ellipses. The left ellipse is larger and more horizontally oriented, while the right one is smaller and more vertically oriented. A vertical blue line passes through the center of the larger ellipse, extending above and below its vertical extent.



Example

$\sigma_2:$

$g_1^{\sigma_2} = g_1$

A diagram illustrating a mathematical concept. It features a vertical blue line. Two overlapping green ellipses are drawn, with the vertical line passing through the center of the larger, outer ellipse. The smaller, inner ellipse is positioned to the right of the vertical line, overlapping with the right side of the larger ellipse.



Example

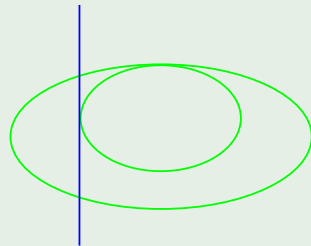
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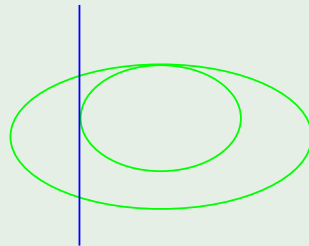
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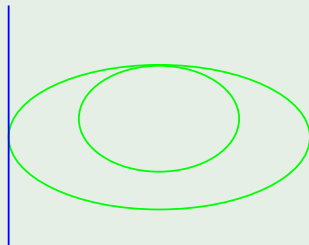
$\sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_3$ :

The diagram consists of a light green rectangular area. On the left side, there is a vertical blue line. To the right of this line, there are two concentric green ellipses. The inner ellipse is smaller and more circular, while the outer ellipse is larger and more elongated horizontally. The blue line is positioned to the left of the inner ellipse, touching its left edge.

Example

$\sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_3$ :

$$g_1^{\sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_3} = g_2^{-1} g_4 g_2 \quad \Rightarrow \quad g_4 = g_2 g_1 g_2^{-1}$$

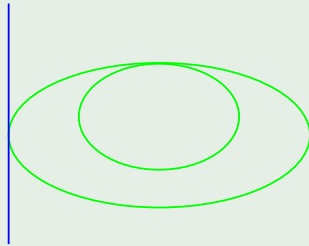


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$$g_2^{\sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_3} = g_2$$



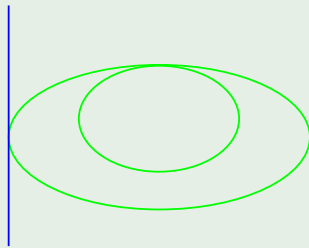
### Example

$\sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_3$ :

$$g_1^{\sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_3} = g_2^{-1} g_4 g_2 \quad \Rightarrow g_4 = g_2 g_1 g_2^{-1}$$

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### Example

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$$g_3^{\sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_3} = g_3$$

$$g_4^{\sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_3} = g_4 g_2 g_1 g_2^{-1} g_4^{-1} \quad \Rightarrow g_4 = g_2 g_1 g_2^{-1}$$

