Zariski-Van Kampen Method

Purpose:
Obtain a presentation for the fundamental group of the complement of a plane projective curve in $\mathbb{P}^2$.
We will put together several ingredients, among which, the *Van Kampen Theorem* is key.

Let $\pi : X \longrightarrow M$ be a locally trivial fibration with section $s : M \to X$. Consider $p \in M$ and $x_0 \in F_p$.

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Meridians around the same irreducible components of B are conjugate in $\pi_1(M \setminus B)$. Moreover, the conjugacy class of a meridian coincides with the set of homotopy classes of meridians around the same irreducible component.

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Proposition

The inclusion $M \setminus B \hookrightarrow M$ induces a surjective morphism, whose kernel is given by the smallest normal subgroup of $\pi_1(M \setminus B)$ containing meridians of all the irreducible components of B.

Zariski-Van Kampen Theorem

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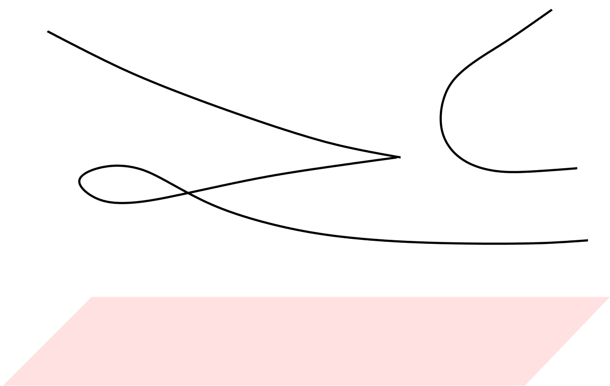

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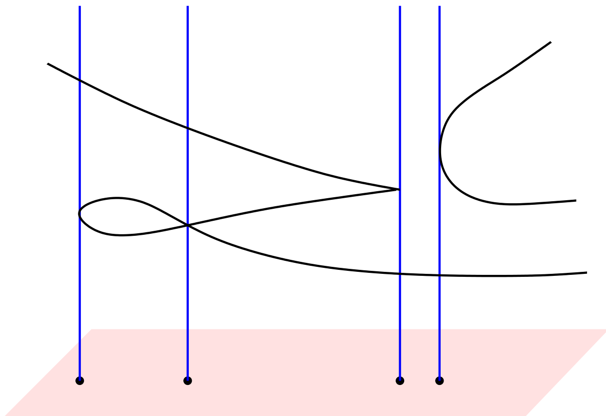

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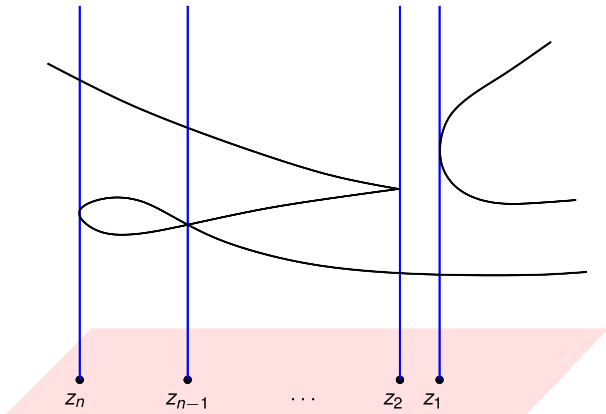

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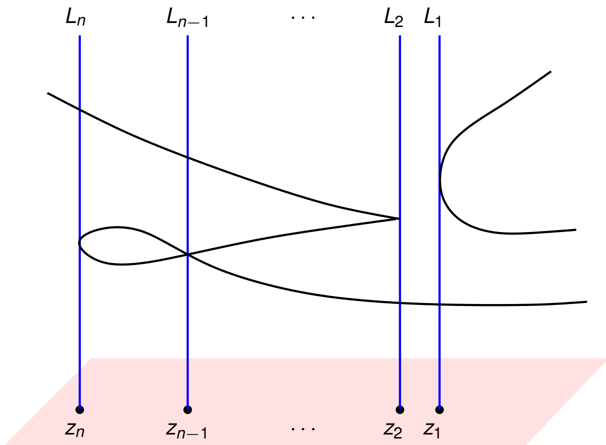

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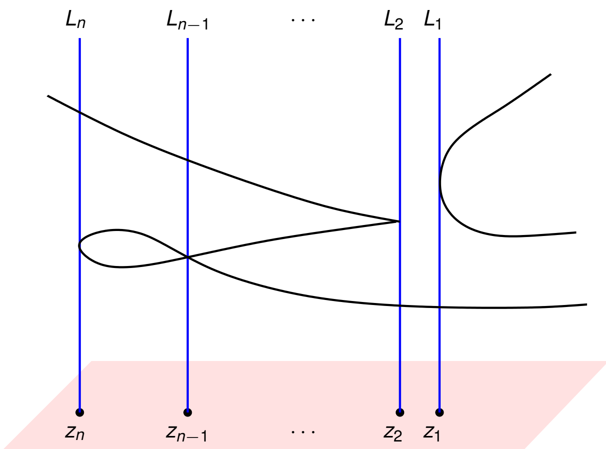

Remark (1)

Let $X = \mathbb{P}^2 \setminus (\mathcal{C} \cup L)$, then $\pi|_X : X \to \mathbb{P}^1 \setminus Z_n$ is a locally trivial fibration.

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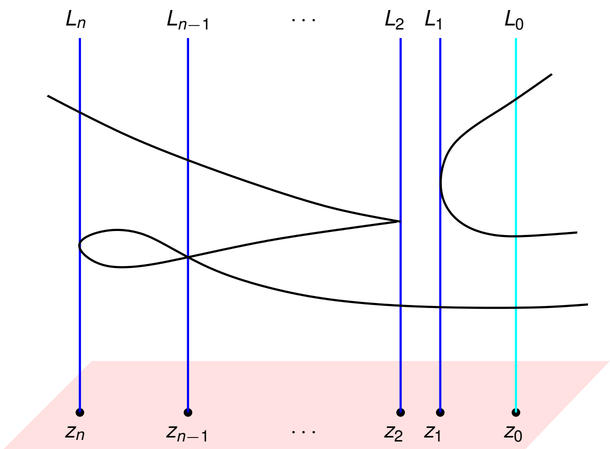
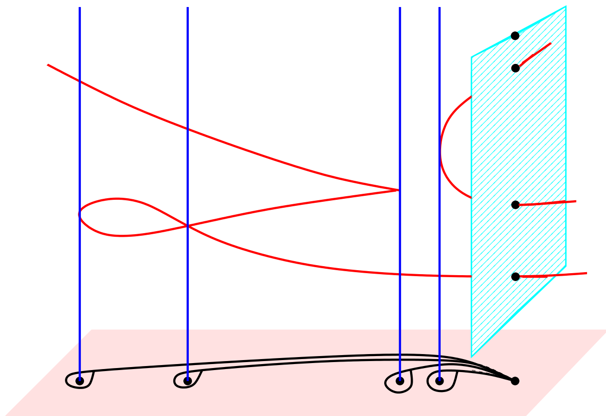


Remark (1)

Let $X = \mathbb{P}^2 \setminus (\mathcal{C} \cup L)$, then $\pi|_X : X \to \mathbb{P}^1 \setminus Z_n$ is a locally trivial fibration. Moreover, its fiber is $\mathbb{P}^1 \setminus Z_d$, where $d := \deg \mathcal{C}$.

Zariski-Van Kampen Theorem



Remark (2)

By (2.1), $\pi_1(X, x_0) = \pi_1(F_{z_0}, x_0) \rtimes \pi_1(\mathbb{P}^1 \setminus Z_n, z_0)$. Action is given by the monodromy of $\pi_1(\mathbb{P}^1 \setminus Z_n, z_0)$ on $\pi_1(F_{z_0}, x_0)$.

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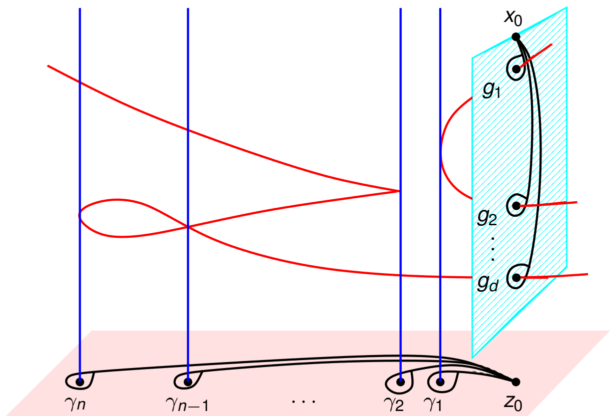
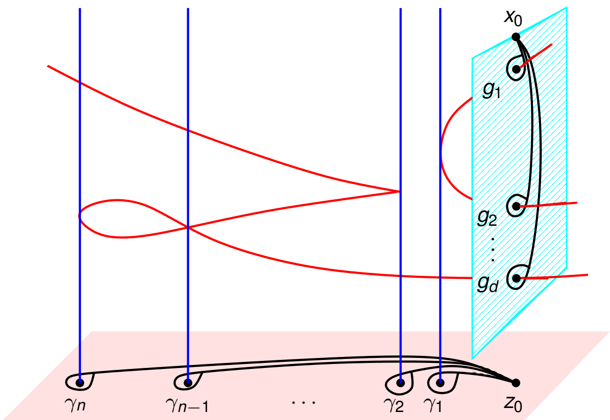


Remark (3)

Note that $\pi_1(F_{z_0}, x_0) = \langle g_1, ..., g_d : g_d g_{d-1} \cdots g_1 = 1 \rangle$ and $\pi_1(\mathbb{P}^1 \setminus Z_n, z_0) = \langle \gamma_1, ..., \gamma_n : \gamma_n \cdots \gamma_1 = 1 \rangle$.

Zariski-Van Kampen Theorem



Theorem

$\pi_1(X, x_0)$ admits the following presentation:

$$\langle g_1, ..., g_d, \gamma_1, ..., \gamma_n : g_d g_{d-1} \cdots g_1 = \gamma_n \cdots \gamma_1 = 1, g_i^{\gamma_j} = \gamma_j^{-1} g_i \gamma_j \rangle$$

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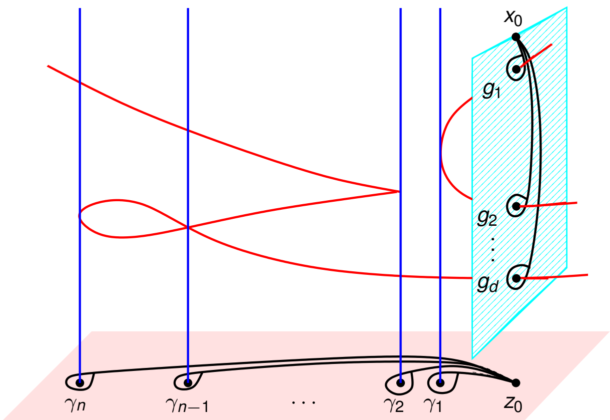

Theorem

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ admits the following presentation:

$$\langle g_1, \ldots, g_d : g_d g_{d-1} \cdots g_1 = 1, g_i^{\gamma_j} = g_i \rangle$$

Remark

- Let $\mathcal{C} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_r$ the decomposition of $\mathcal{C}$ in its irreducible components, then

$$H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/(d_1, \ldots, d_r),$$

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$$H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/(d_1, ..., d_r),$$

  where $d_i := \deg \mathcal{C}$.
- It two curves are in a connected family of equisingular curves, then they are isotopic

Zariski-Van Kampen Theorem

Example

$\mathcal{C}$ smooth of degree $d \Rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$.

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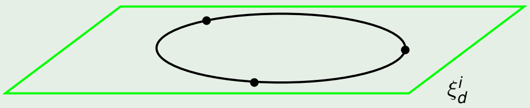
Let us compute the local monodromy of $x = y^d$. Consider $\gamma(t) = e^{2\pi t\sqrt{-1}}$ a loop around $x = 0$. The fiber at $\gamma(t)$ is given by:



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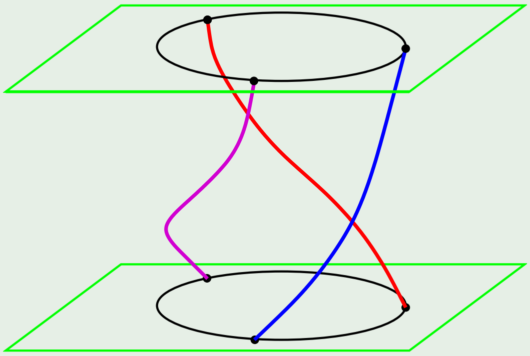
The monodromy around $x = 0$ looks as follows:



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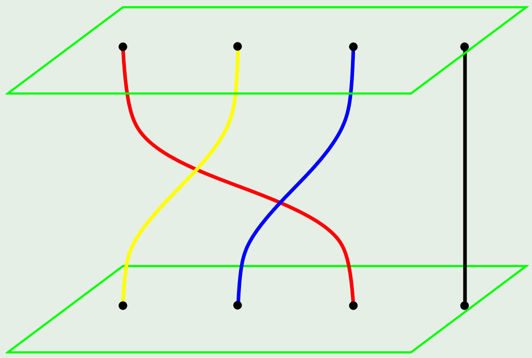
$\mathcal{C}$ smooth of degree $d \Rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$.
Corresponds to the braid $\sigma_1\sigma_2 \cdots \sigma_{d-1}$



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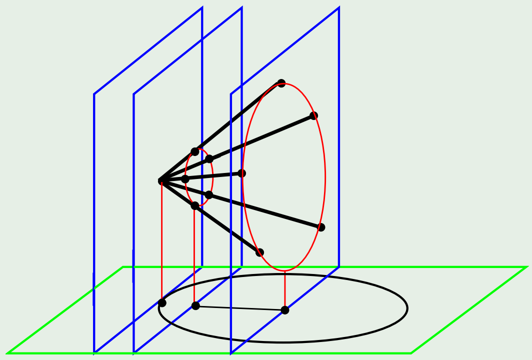
Note that the global part of the monodromy has no contribution:



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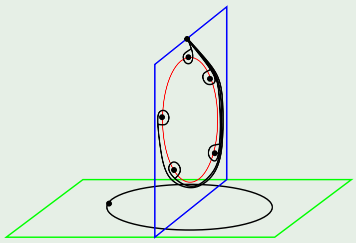

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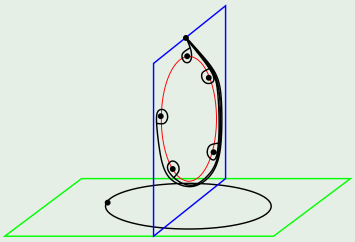

One obtains:

$$g_i = g_i^{(\sigma_1\sigma_2\cdots\sigma_{d-1})} = \begin{cases} g_d & i = 1 \\ g_d^{-1} g_{i-1} g_d & i \neq 1 \end{cases}$$

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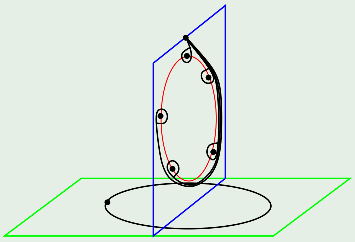

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hence $g_2 = g_d^{-1} g_1 g_d = g_1$, and by induction $g_1 = ... = g_d = g$. Finally, $g_1 \cdots g_d = 1$ becomes $g^d = 1$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \langle g : g^d = 1 \rangle = \mathbb{Z}/d\mathbb{Z}.$$

Example (Zariski-Harris, Cheniot)

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Example (Zariski)

Let $\mathcal{C}$ be a general nodal rational curve of degree d. Consider $\tilde{\mathcal{C}}$ its dual. Note that $\tilde{\mathcal{C}}$ is a rational curve of degree $2(d-1)$, $2(d-2)(d-3)$ nodes, and $3(d-2)$ cusps.
The fundamental group of $\tilde{\mathcal{C}}$ coincides with the fundamental group of the unordered configuration space of d points in $\mathbb{S}^2$, that is,

$$\mathbb{B}_d(\mathbb{S}^2) = \left\langle g_1, ..., g_{d-1} : \begin{array}{l} g_i g_j = g_j g_i, \\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \\ g_1 \cdots g_{d-2} g_{d-1}^2 g_{d-2} \cdots g_1 = 1 \end{array} \right\rangle.$$

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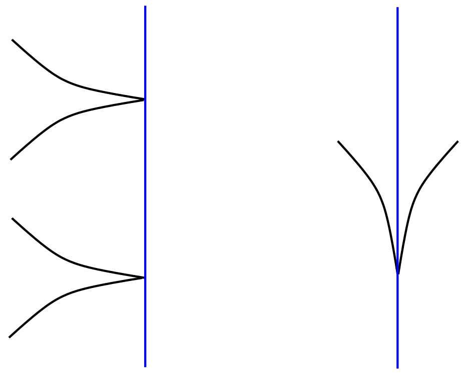

Local Braid Monodromy

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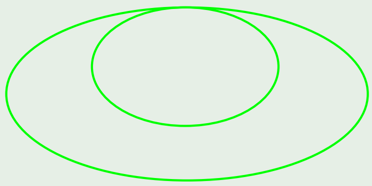
- Computational methods are effective essentially over $\mathbb{Z}[\sqrt{-1}]$.

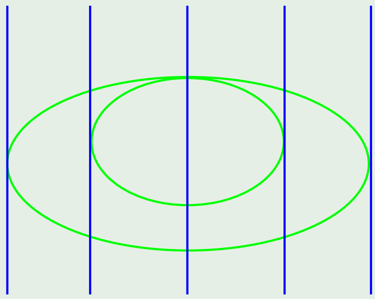
Example

Consider the following quartic:



Example

Consider the following quartic and project from $[0:1:0]$



Example

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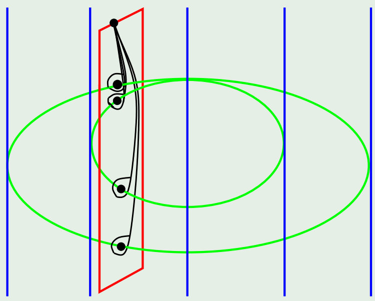

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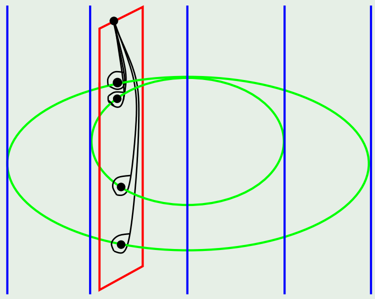
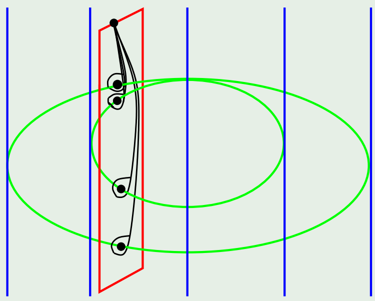


Example

Compute the braid monodromy: σ_1^8, σ_2,



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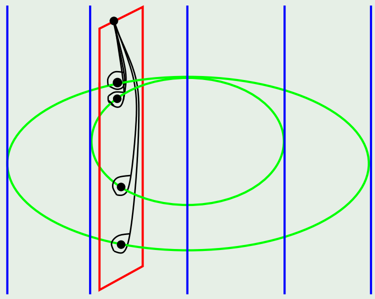

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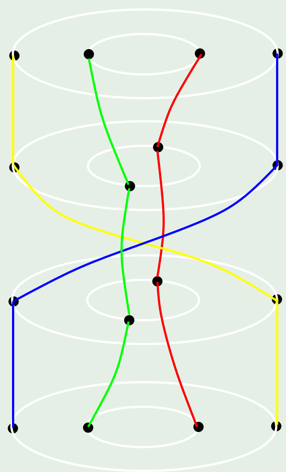

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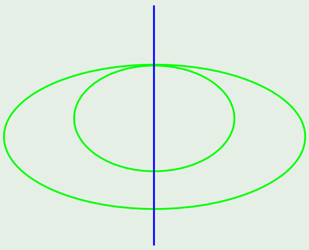
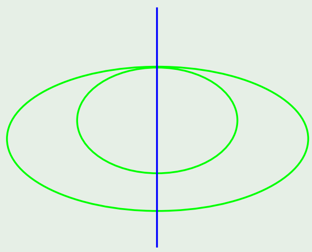


Example

σ_1^8:

$$g_1^{\sigma_1^8} = (g_2 g_1)^4 g_1 (g_2 g_1)^{-4} \quad \Rightarrow [(g_2 g_1)^4, g_1] = 1$$



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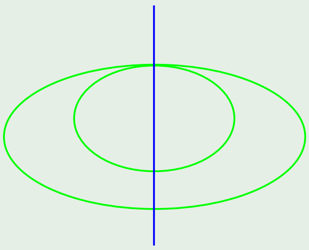

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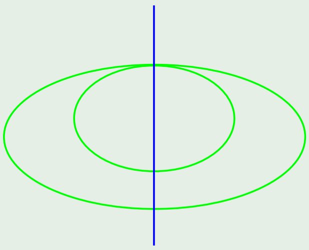

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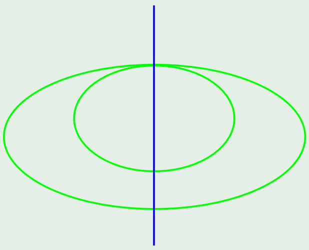

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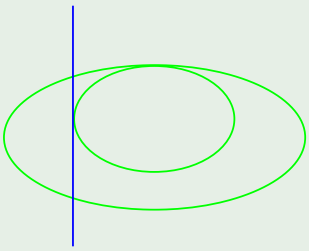

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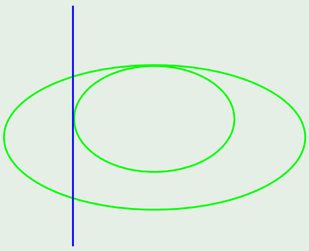
σ_2:

$$g_1^{\sigma_2} = g_1$$



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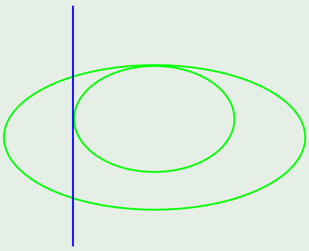
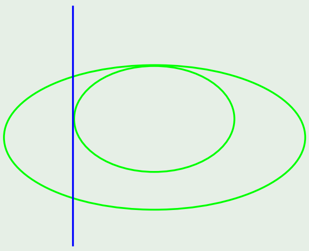


Example

σ_2:

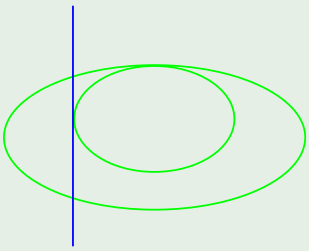
$$\begin{aligned} g_1^{\sigma_2} &= g_1 & \\ g_2^{\sigma_2} &= g_3 & \Rightarrow g_2 = g_3 \\ g_3^{\sigma_2} &= g_3 g_2 g_3^{-1} & \Rightarrow g_2 = g_3 \end{aligned}$$



Example

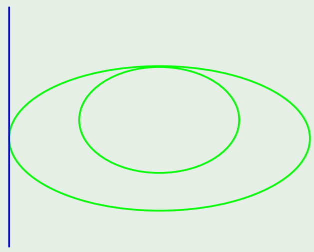
σ_2:

$$\begin{aligned} g_1^{\sigma_2} &= g_1 & \\ g_2^{\sigma_2} &= g_3 & \Rightarrow g_2 = g_3 \\ g_3^{\sigma_2} &= g_3 g_2 g_3^{-1} & \Rightarrow g_2 = g_3 \\ g_4^{\sigma_2} &= g_4 & \end{aligned}$$



Example

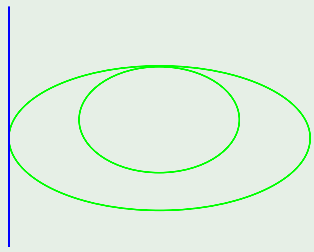
$\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3$:



Example

$\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3$:

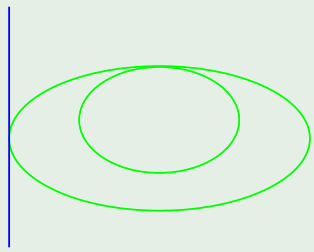
$$g_1^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} = g_2^{-1}g_4g_2 \qquad \Rightarrow g_4 = g_2g_1g_2^{-1}$$



Example

$\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3$:

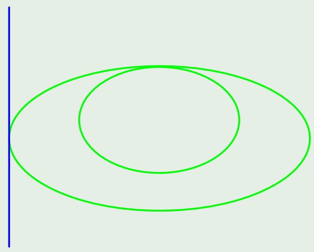
$$g_1^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} = g_2^{-1}g_4g_2 \qquad \Rightarrow g_4 = g_2g_1g_2^{-1}$$
$$g_2^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} = g_2$$



Example

$\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3$:

$$g_1^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} = g_2^{-1}g_4g_2 \qquad \Rightarrow g_4 = g_2g_1g_2^{-1}$$
$$g_2^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} = g_2$$
$$g_3^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} = g_3$$



Example

$\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3$:

$$\begin{aligned}
g_1^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} &= g_2^{-1}g_4g_2 && \Rightarrow g_4 = g_2g_1g_2^{-1} \\
g_2^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} &= g_2 \\
g_3^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} &= g_3 \\
g_4^{\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3} &= g_4g_2g_1g_2^{-1}g_4^{-1} && \Rightarrow g_4 = g_2g_1g_2^{-1}
\end{aligned}$$