Braid Monodromy Of Algebraic Plane Curves

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  such that:
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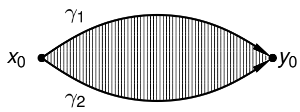

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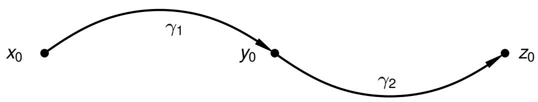

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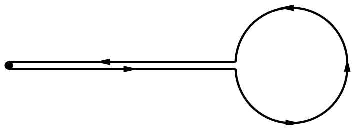

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- X connected $\Rightarrow \pi_1(X)$

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Example (Ordered Configuration Spaces)

Let $X_n := \{(z_1, ..., z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}$. Then $\pi_1(X_n) = \mathbb{P}_n$.

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Example (Non-ordered Configuration Spaces)

Let $\mathcal{P}_n := \{f(z) \in \mathbb{C}[z] \mid \deg(f) = n\}$, $Y_n := \mathbb{P}(\mathcal{P}_n \setminus \Delta_n)$, where $\Delta_n := \{f \in \mathcal{P}_n \mid f \text{ has multiple roots}\}$. Note that $Y_n \cong X_n/\Sigma_n$. Then $\pi_1(Y_n) = \mathbb{B}_n$. Analogously, if we consider $\bar{\mathcal{P}}_n := \{f(s,t) \in \mathbb{C}[s,t] \mid f \text{ homogeneous } \deg(f) = n\}$, $\bar{Y}_n := \mathbb{P}(\mathcal{P}_n \setminus \Delta_n)$, where $\bar{\Delta}_n := \{f \in \bar{\mathcal{P}}_n \mid f \text{ has multiple roots}\}$. Note that $\pi_1(\bar{Y}_n) = \mathbb{B}_n(\mathbb{S}^2)$.

Van Kampen Theorem

Theorem

Let U_1 and U_2 open subsets of X such that:

- $U_1 \cup U_2 = X$ *and*
- $U_{12} := U_1 \cap U_2$ *is path-connected.*

Then

$$\pi_1(X) = \pi_1(U_1) *_{\pi_1(U_{12})} \pi_1(U_2).$$

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Let $z_1, ..., z_n \in \mathbb{C}$, $Z_n := \{z_1, ..., z_n\}$. Then $\pi_1(\mathbb{C} \setminus Z_n) = \mathbb{F}_n$.

Locally trivial Fibrations

Definition

A surjective smooth map $\pi : X \longrightarrow M$ of smooth manifolds is a *locally trivial fibration* if there is an open cover $\mathcal{U}$ of M and diffeomorphisms $\varphi_U : \pi^{-1}(U) \longrightarrow U \times \pi^{-1}(p_U)$, with $p_U \in U$, such that φ_U is fiber-preserving, that is $pr_1 \varphi_U = \pi$. We denote $\pi^{-1}(p)$ by F_p.

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Consider $\pi : X \longrightarrow M$ a locally trivial fibration and $s : M \longrightarrow X$ a *section*. There is an action of $\pi_1(M, p)$ on $\pi_1(F_p, x_0)$ ($s(p) = x_0$) called *monodromy action of* M on F_p.

Monodromy Actions

$$\begin{array}{rccc} \pi^{-1}(\gamma) = & \tilde{X} & \hookrightarrow & X \\ & \downarrow \tilde{\pi} & & \downarrow \pi \\ & I & \xrightarrow{\gamma} & M \end{array}$$

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The fibration $\tilde{\pi}$ is trivial, and hence there exists

$$\varphi : I \times F_p \to \tilde{X}$$

such that $\varphi(0, x) = Id_{F_p}$.
If π is such that F_p is connected, then given a loop $\alpha \in \pi_1(F_p, x_0)$ and a loop $\gamma \in \pi_1(M, p)$, then one deforms $\varphi(t, \alpha)$ into a loop $\alpha_t \in \Gamma(F_{\gamma(t)}, s(\gamma(t)))$. Then $\alpha^\gamma := \alpha_1$ is the monodromy action of γ over α.

Remark

Another interesting scenario occurs when F_p is finite and π is a topological cover. In that case $\varphi(1, x)$ induces a permutation of F_p. This permutation is also called the *monodromy action of γ over F_p*.

Examples

Example

Let $\pi : X = M \times F \to M$ be a trivial fibration. Any continuous map $\omega : M \to F$, defines $s(x) = (x, \omega(x))$ a section of $\pi : X \to M$. In this case, φ is the identity. Let $\gamma \in \pi_1(M, p)$ and $\alpha \in \pi_1(F, x_0)$, then α_t is given by $(\omega_t \circ \gamma)^{-1}\alpha(\omega_t \circ \gamma)$, where $\omega_t \circ \gamma(\lambda) = \omega(\gamma(\lambda t))$. Therefore $\pi_1(M, p)$ acts on $\pi_1(F, \omega(p))$ by

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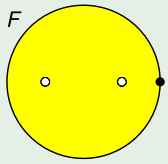

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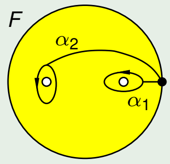

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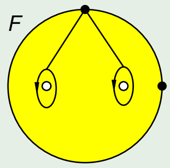

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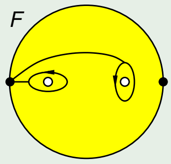

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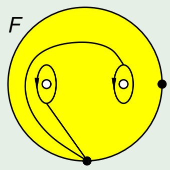

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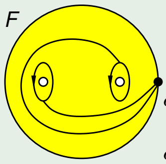

$$\alpha_1^\gamma = (\alpha_2\alpha_1)^{-1}\alpha_1(\alpha_2\alpha_1)$$

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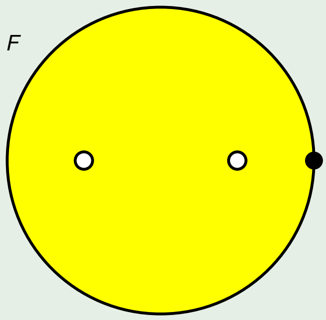

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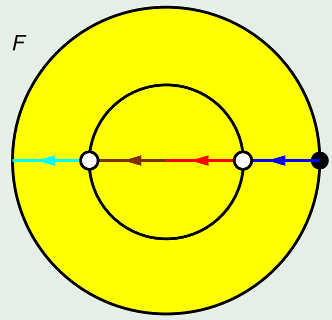

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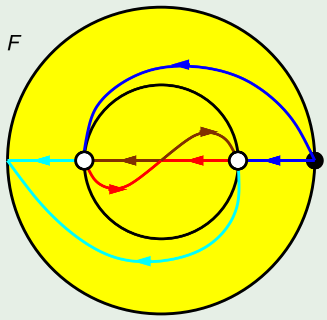

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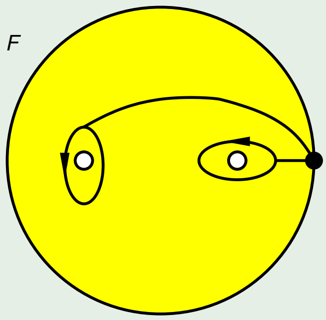

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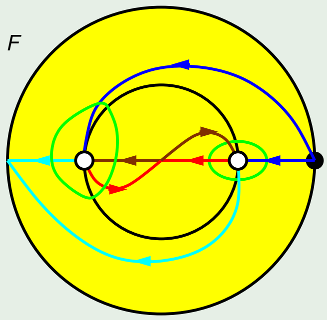

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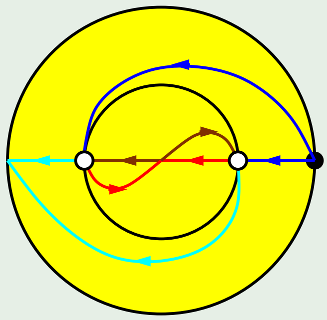

$$\alpha_1^\gamma = \alpha_2$$

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Mapping Class Group

Theorem

There is an isomorphism between the geometric group of braids on n-strings and the mapping class group of automorphisms on the punctured disc $\mathbb{D}_n := \mathbb{D} \setminus Z_n$ *modulo homotopy relative to the boundary, that is,* $\pi_0(Diff^+(X_n))$.

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$$g_j^{\sigma_i} = \begin{cases} g_{i+1} & j = i \\ g_{i+1} g_i g_{i+1}^{-1} & j = i+1 \\ g_i & \text{otherwise.} \end{cases}$$

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- Since $(g_n \cdot \dots \cdot g_1) = \partial\mathbb{D}$, one obtains $(g_n \cdot \dots \cdot g_1)^\sigma = (g_n \cdot \dots \cdot g_1)$.

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Definition

Let M be an m-dimensional (connected) complex manifold. A *branched covering* of M is an m-dimensional irreducible normal complex space X together with a surjective holomorphic map $\pi : X \longrightarrow M$ such that:

- every fiber of π is discrete in X,
- $R_\pi := \{q \in X \mid \pi^* : \mathcal{O}_{\pi(q),M} \longrightarrow \mathcal{O}_{q,X} \text{is not an isomorphism}\}$ called the *ramification locus*, and $B_\pi = \pi(R_\pi)$ called the *branched locus*, are hypersurfaces of X and M, resp.
- $\pi| : X \setminus \pi^{-1}(B_\pi) \longrightarrow M \setminus B_\pi$ is an unramified (topological) covering, and
- $\forall\, p \in M$ there is a connected open neighborhood $W^p \subset M$ such that for every connected component U of $\pi^{-1}(W)$:
  i) $\pi^{-1}(p) \cap U = \{q\}$
  ii) $\pi|_U : U \longrightarrow W$ is surjective and proper.

Construction of branched coverings: smooth case

If B is a non-singular hypersurface, $B = D_1 \cup \ldots \cup D_n$, $e_1, \ldots, e_n \in \mathbb{N}$, $D = \sum e_i D_i$ on M.
$p_0 \in M \setminus B$ base point.

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Condition

If $\gamma_j^d \in J$ then $d \equiv 0 \ (mod\ e_j) \ \forall\ 1 \leq j \leq s$.

Theorem

There is a natural one-to-one correspondence between

$$\{\pi : X \to M \text{ Galois, finite, ramified along } D\} / \sim$$
$$\updownarrow$$
$$\{J \subset K \overset{f.i}{\triangleleft} \pi_1(M \setminus B) \text{ satisfying (1.4)}\}.$$

Moreover, there is a maximal Galois covering $\pi(M, D)$ of M ramified along D iff $K_\pi = \cap K \overset{f.i}{\triangleleft} \pi_1(M \setminus B)$ satisfies (1.4).

Construction of branched coverings: smooth case

Theorem (Riemann Existence Theorem)

Any monodromy action $\pi_1(\mathbb{P}^1 \setminus Z_n) \to \Sigma_s$ *can be realized by a branched covering of the projective line* $\mathbb{P}^1$.

Construction of branched coverings: general case

If B is a hypersurface, $B = D_1 \cup ... \cup D_n$, $e_1, ..., e_n \in \mathbb{N}$, $D = \sum e_i D_i$ on M. $p_0 \in M \setminus B$ base point.

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$i : W^{\tilde{p}_0} \setminus B \hookrightarrow M \setminus B$.

Condition

Let $K \triangleleft \pi_1(M \setminus B, p_0)$ such that $J \triangleleft K$. For any point $p \in \operatorname{Sing} B$,
$K_p = i_*^{-1}(K) \overset{f.i}{\triangleleft} \pi_1(W \setminus B, \tilde{p})$.

Theorem

There is a one-to-one correspondence:

$$\{\pi : X \to M \text{ Galois, finite, ramified along } D\} / \sim$$
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$$\{J \subset K \overset{f.i}{\triangleleft} \pi_1(M \setminus B) \text{ satisfying (1.4) and (1.7)}\}.$$

Moreover, there is a maximal Galois covering $\pi(M, D)$ of M ramified along D iff $K_\pi = \cap K \overset{f.i}{\triangleleft} \pi_1(M \setminus B)$ satisfies (1.4) and (1.7).

Example

Consider $M = \mathbb{P}^2$, $D_1 = \{zy^2 = x^3\}$, $D_2 = \{z = 0\}$. Let us study the possible Galois covers of $\mathbb{P}^2$ ramified along $D = e_1 D_1 + e_2 D_2$.

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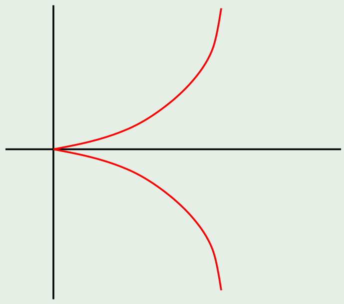

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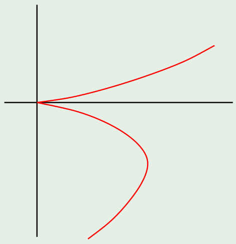

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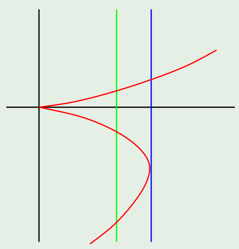

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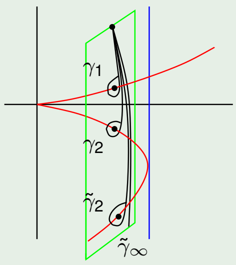

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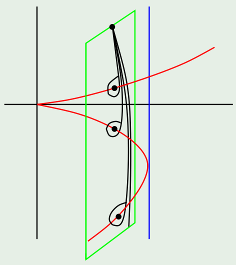

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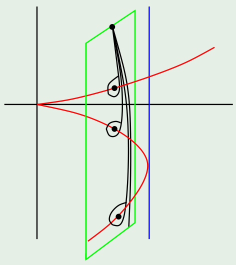

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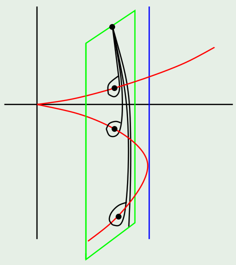

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$$\gamma_2 \gamma_1 \gamma_2 = \gamma_1 \gamma_2 \gamma_1.$$

Theorem

In the following cases there is a maximal Galois covering of $\mathbb{P}^2$ ramified along D:

(e_1, e_2)	$G = \pi_1(\mathbb{P}^2 \setminus D)/J$	$\|G\|$
$(2, 2)$	Σ_3	*6*
$(3, 4)$	$SL(2, \mathbb{Z}/3\mathbb{Z})$	*24*
$(4, 8)$	$\Sigma_4 \ltimes \mathbb{Z}/4\mathbb{Z}$	*96*
$(5, 20)$	$SL(2, \mathbb{Z}/5\mathbb{Z}) \times \mathbb{Z}/5\mathbb{Z}$	*600*

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However, there is no maximal Galois cover of $\mathbb{P}^2$ ramified along $D = 6D_1 + 2D_2$.

Theorem

Let $B = D_1 \cup ... \cup D_n$. Then any representation of $\pi_1(M \setminus B)$ on a linear group $GL(r, \mathbb{C})$ such that the image of a meridian γ_i has order e_i, gives rise to a Galois cover of M branched along $D = e_1 D_1 + ... + e_n D_n$.

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Theorem (Hamm,Goreski-MacPherson)

Let $M \subset \mathbb{P}^n$ be a closed subvariety which is locally a complete intersection of dimension m. Let $\mathcal{A}$ be a Whitney stratification of M and consider $B \subset \mathbb{P}^n$ another subvariety such that $B \cap M$ is a union of strata of $\mathcal{A}$. Consider H a hyperplane transversal to $\mathcal{A}$ in $M \setminus B$, then the inclusion

$$(M \setminus B) \cap H \hookrightarrow M \setminus B$$

is an $(m-1)$-homotopy equivalence.

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- Chisini Problem:
  Let S be a nonsingular compact complex surface, let $\pi : S \to \mathbb{P}^2$ be a finite morphism having simple branching, and let B be the branch curve; then "to what extent does the pair $(\mathbb{P}^2, B)$ determine π"?