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Hilden braid groups

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joint work with

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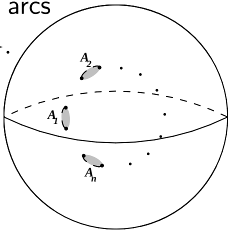
Hilden groups

Let $\mathcal{A}_n = A_1 \cup \cdots \cup A_n$ be system of *trivial* arcs
and $\mathcal{P}_{2n} = \partial(\mathcal{A}_n) = \{P_{i,1}, P_{i,2} \mid i = 1, \ldots, n\}$.



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The *Hilden group* Hil_n on n arcs is the group $\mathrm{im} R_n$.

Theorem[Hilden, 1975] A set of generators for Hil_n is given by

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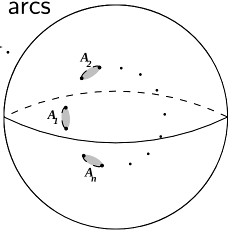

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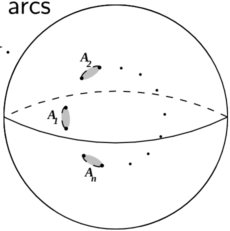

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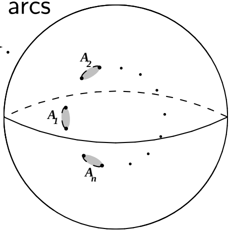

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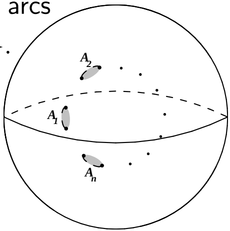

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Using this presentation it is possible to obtain a presentation for Hil_n adding the relations corresponding to the kernel of the surjection from $\mathsf{PMCG}_{2n}(\mathbf{D}^2) \longrightarrow \mathsf{PMCG}_{2n}(\mathbb{S}^2)$.

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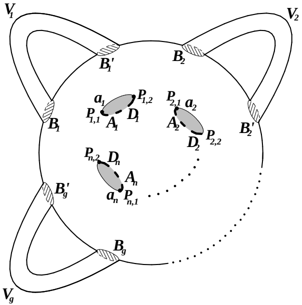
A generalization: Hilden braid groups

Let H_g be a genus g handlebody and $T_g = \partial H_g$. As before, let $\mathcal{A}_n$ be system of trivial arcs and $\mathcal{P}_{2n} = \partial(\mathcal{A}_n) \subset T_g$. Consider

$$\begin{array}{ccc} \mathrm{MCG}_n(H_g) & \xrightarrow{\bar{\Omega}_{g,n}} & \mathrm{MCG}(H_g) \\ \downarrow R_{g,n} & & \downarrow R_{g,0} \\ \mathrm{MCG}_{2n}(T_g) & \xrightarrow{\Omega_{g,n}} & \mathrm{MCG}(T_g). \end{array}$$



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Theorem[Birman, 1974] If $g \geq 2$, the group $\ker \Omega_{g,n}$ is isomorphic to $B_{2n}(T_g)$.

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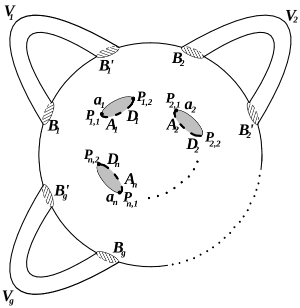

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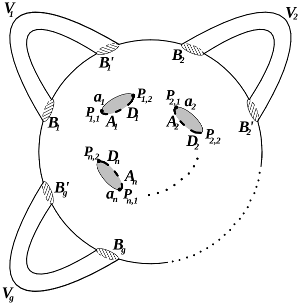

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Generalized plat closure

Let M be a closed, connected, orientable 3-manifold and let $\psi \in \mathrm{MCG}(\mathrm{T}_{g,1})$ be a fixed element such that

$$M = \mathrm{H}_g \cup_{\tau\psi_0} \bar{\mathrm{H}}_g$$

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Recall that $\Omega_{n,g} : \mathrm{MCG}_{2n}(\mathrm{T}_g) \to \mathrm{MCG}(\mathrm{T}_g)$.

The *generalized plat closure* of the couple (M, ψ) is

$$\Theta^{\psi}_{g,n} : \ker \Omega_{g,n} \longrightarrow \{\text{links in } M\} \qquad \Theta^{\psi}_{g,n}(\sigma) = \hat{\sigma}^{\psi}$$

where

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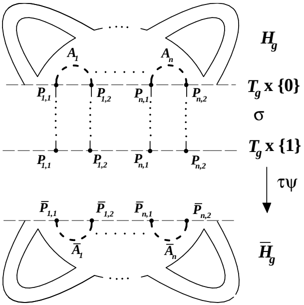

Proposition[Bellingeri, C.] For each link L in M there exist $n \in \mathbb{N}$ and $\sigma \in \ker \Omega_{g,n}$ such that $L = \hat{\sigma}^{\psi}$. Moreover

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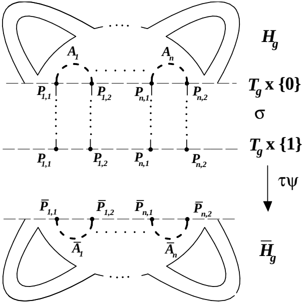
Open problem Find the equivalence moves under generalized plat closure.



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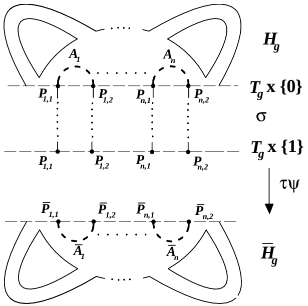

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Motion groups

A *motion* of a submanifold N in a closed manifold M is a path f_t in $\mathrm{Homeo}(M)$ such that $f_0 = \mathrm{id}_M$ and $f_1(N) = N$. A motion is called *stationary* if $f_t(N) = N$ for all $t \in [0,1]$. The *motion group* $\mathcal{M}(M,N)$ of N in M is the group of equivalence classes of motion of N in M where two motions f_t, g_t are equivalent if $(g^{-1}f)_t$ is homotopic, relative to endpoints, to a stationary motion.

Generators for the motion group of the n-component trivial link and all the torus links in $\mathbb{S}^3$ can be found in [Goldsmith, 1981-1982].

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Corollary Let $M = \mathbb{S}^3$. The homomorphism $\mathcal{H}_\psi : \mathrm{Hil}_n^g \cap \mathrm{Hil}_n^g(\psi) \to \mathcal{M}(\mathbb{S}^3, L_n)$ is surjective. Moreover, it is injective if and only if $(g,n) = (0,1)$.
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Open problem Find generators for the motion groups of a links in a 3-manifold different from $\mathbb{S}^3$.

GENERATORS OF Hil_n^g

THEOREM[Bellingeri, C.] The group Hil_n^g is generated by

1) the twist of the fist arc and the exchange of the j-th and $(j+1)$-th arcs with $j = 1, \ldots, n-1$;
2) the *slides* of the first arc along the curves $\mu_{1,j}$, $\lambda_{1,j}$, $\sigma_{1,r}$ with $k = 1, \ldots, g$ and $r = 1, \ldots n$;
3) all the *admissible* slides of the meridian discs.

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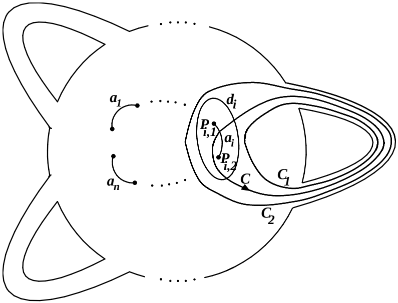

The slide $S_{i,C} = T_{C_1}^{-1} T_{C_2} s_i$ of the i-th arc along the curve C.

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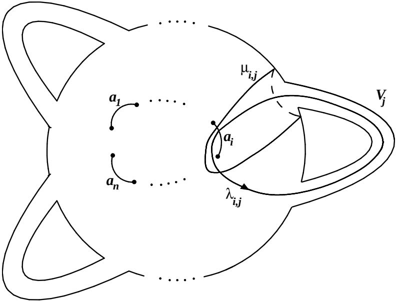

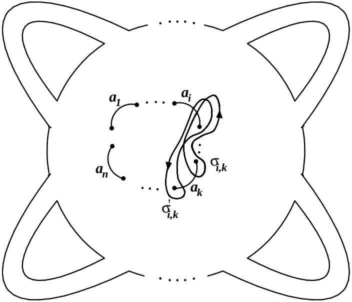

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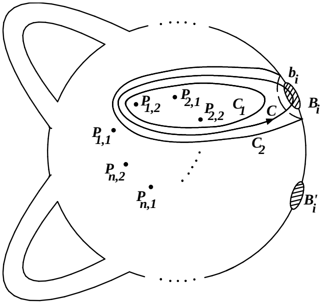

The slide $M_{i,C} = T_{C_1}^{-1} T_{C_2} T_{b_i}^{-1}$ of the meridian disk B_i along the curve C.

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If $g = 1$ all the sliding curves for the meridian discs are admissible, so Hil_n^1 is finitely generated.

OPEN PROBLEM Is Hil_n^g finitely generated for $g \geq 2$?

Thanks!

Gracias!

cπacнσo!

Grazie!

Merci!

Danke!