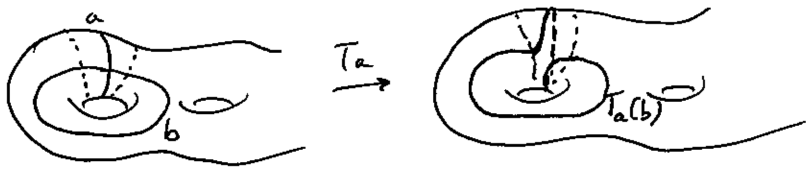
I Theorem

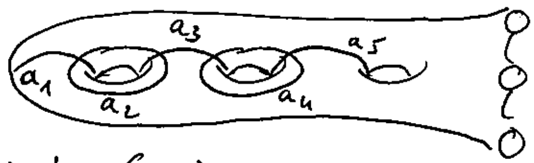
- Σ = surface genus g, b boundary components
- $PDiff^+(\Sigma) := \{$ diffeomorphisms, $\oplus$, that preserve each boundary component $\}$
- $Diff_0(\Sigma) := \{$ diff. $\simeq$ Identity $\} \lhd PDiff^+(\Sigma)$
- $PMod(\Sigma) := PDiff^+(\Sigma) / Diff_0(\Sigma)$
- Dehn twist T_a about a curve a:



Basic Facts about Dehn twists:

- $a \cap b = \emptyset \iff T_a T_b = T_b T_a$
- $a \cap b = \{*\} \iff T_a T_b T_a = T_b T_a T_b$

Construction: $n \geq 6$, even, $g \geq \frac{n}{2} - 1$, $b \geq 0$. Σ



Let $(a_i)_{i \leq n-1}$ be a $(n-1)$-chain of curves.

Let φ: $\mathbb{F}_{n-1} \longrightarrow PMod(\Sigma)$
$x_i \longrightarrow T_{a_i}$

$$\operatorname{Ker} \varphi \supseteq \operatorname{Norm}\left(\left\{\begin{array}{ll} x_i x_j = x_j x_i, & |i-j| \geq 2 \\ x_i x_j x_i = x_j x_i x_j, & |i-j| = 1 \end{array}\right\}\right)$$

Let

$$\left.\begin{array}{lll} \rho_0: & B_n & \longrightarrow PMod(\Sigma) \\ & \tau_i & \longmapsto T_{a_i} \\ \rho_1: & \tau_i & \longmapsto T_{a_i}^{-1} \\ \rho_2: & \tau_i & \longmapsto T_{a_i}^{\varepsilon} V \end{array}\right\} \text{monodromy morphism}$$

V s.t. $\forall i$, $V \rightleftarrows A_i$

Theorem: $\boxed{\begin{array}{l} n \geq 6,\ g \leq \frac{n}{2},\ b \geq 0 \\ \text{Any morphism } \rho \text{ from } B_n \text{ to } PMod(\Sigma) \\ \text{is cyclic or is a monodromy morphism} \end{array}}$

cyclic: $\rho(\tau_1) = \rho(\tau_2) = \rho(\tau_3) = \dots$

II Beginning of the proof (Case of irreducible el^{ts})

Let $\rho : \begin{array}{l} B_n \longrightarrow PMod(\Sigma) \\ \tau_i \longmapsto A_i := \rho(\tau_i) \end{array}$ with $g \leq \frac{n}{2}$, <u>non cyclic</u>

Is $A_i = T_{a_i}^{\varepsilon} V$? Why is A_i neither **periodic** nor **pseudo-Anosov**

$\boxed{\forall a \in Curve(\Sigma),\ \exists n > 0 \mid F^n(a) = a}$

$\boxed{\forall a \in Curve(\Sigma),\ \forall n > 0,\ F^n(a) \neq a}$

<u>Fact:</u> $\boxed{\text{The } A_i\text{'s are conjugate}}$

<u>Proof:</u> In B_n, $\tau_2 = (\tau_1\tau_2)\tau_1(\tau_1\tau_2)^{-1}$ ∎

<u>Prop 1:</u> $\boxed{\text{If } A_1 \text{ is periodic, then } \rho \text{ is cyclic}}$

<u>Proof:</u> ⊙ Assume $\partial\Sigma \neq \emptyset$. Let d be a boundary.

- $A_1 \rightleftarrows A_4 \Rightarrow \langle A_1, A_4 \rangle$ is a finite group
- $\langle A_1, A_4 \rangle$ preserve $d \Rightarrow$ Th of Riem. Surf.
  $\exists F \mid \langle F \rangle = \langle A_1, A_4 \rangle$
- $m > 0$ smallest s.t. $A_1^m = 1$. So $A_4^m = 1$ and $F^m = 1$.
- So $\langle A_1 \rangle = \langle F \rangle = \langle A_4 \rangle$.
- But (A_1, A_4) is conjugate to (A_2, A_4) with <u>$A_1A_2A_1$</u>
- So $\langle A_1 \rangle = \langle A_4 \rangle \Rightarrow \langle A_2 \rangle = \langle A_4 \rangle$
- Finally $\langle A_1 \rangle = \langle A_2 \rangle = \langle A_3 \rangle = \dots$ <u>ρ cyclic</u> ∎

⊙ If $\partial\Sigma = \emptyset$: use Kerckhoff thm,
find $\mathcal{T}$ finite $\subset \rho(B_n)$ acting freely on points of Σ
use Riemann-Hurewitz formula ($\chi(\Sigma)$, $\chi(\Sigma/_{\mathcal{T}})$)
use $84(g-1)$ theorem.

e.g. if $A_1^2 = Id$, then take $\mathcal{T} = \rho(B_n) \cong \mathfrak{S}_n$
But $|\mathfrak{S}_n| = n! \gg 84(g-1)$ ∎

<u>Prop 2:</u> $\boxed{\text{If } A_1 \text{ is pseudo-Anosov, then } \rho \text{ is cyclic}}$

<u>Proof:</u> • ρ induces $\bar\rho : \begin{array}{l} B_{n-2} \longrightarrow PMod \\ \tau_i \longmapsto \rho(\tau_i\tau_{n-1}^{-1}) = A_iA_{n-1}^{-1} \end{array}$

• $\bar\rho$ is periodic hence cyclic, hence ρ is cyclic ∎

III Following of the proof. Case of reducible elts)

<u>Def:</u> $F \in PMod(\Sigma)$ is said to be <u>reducible</u> if it is neither periodic, nor pseudo-Anosov.

<u>Th:</u> (Nielsen - Thurston)

$$\boxed{\begin{array}{l} \exists\, \sigma : \begin{array}{ccc} PMod(\Sigma) & \longrightarrow & Curve(\Sigma) \\ F & \longmapsto & \sigma(F) \end{array} \quad \text{s.t.} \\ \forall F,\ \exists n > 0 \mid F^n \text{ preserves } \sigma(F), \text{ each component } S \\ \text{of } \Sigma_{\sigma(F)}, \text{ and } \forall S,\ F^n|_S \text{ is pseudo-Anosov or Id.} \end{array}}$$

<u>Properties of σ:</u>

(P1) $\sigma(F) = \emptyset \iff F$ is irreducible

(P2) $\sigma(T_a) = \{a\}$

(P3) $\sigma(FGF^{-1}) = F(\sigma(G))$

non intersecting

(P4) $F \rightleftarrows G \iff \sigma(F) \cup \sigma(G)$ is a set of disjoint curves.

(P5) $F(a) = a,\ a \notin \sigma(F) \implies \sigma(T_a F) = \{a\} \cup \sigma(F)$

<u>Recall:</u>

$$\rho : \begin{array}{ccc} B_n & \longrightarrow & PMod(\Sigma) \\ \sigma_i & \longmapsto & A_i \end{array} \qquad g \leq \frac{n}{2} \quad \underline{\text{non cyclic}}$$

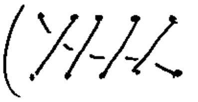
If $A_i = T_{a_i} V$, then $\sigma(A_i) = \{a_i\} \cup \sigma(V)$

Notice that a_i is specific to A_i:

$$a_i \in \sigma(A_i) \text{ but } a_i \notin \sigma(A_j) \quad \forall j \neq i$$

Why do specific curves exist (like a_i)?

<u>Prop 3:</u> $\boxed{\text{Because actually, } \exists\, a_i \in \sigma(A_i) \mid a_i \cap D(a_i) \neq \emptyset}$

where $D = A_1 A_2 \cdots A_{n-1} = \rho\Big($  $\Big)$

"because" for
- $a_i \in \sigma(A_i) \implies D(a_i) \in \sigma(D A_i D^{-1}) = \sigma(A_{i+1})$
- a_i meets a curve in $\sigma(A_{i+1})$ so:
  $a_i \in \sigma(A_j) \implies j = i$ or $i+2$
- $a_i \cap D(a_i) \neq \emptyset \iff D^{-1}(a_i) \cap a_i \neq \emptyset$
- Finally $a_i \in \sigma(A_j) \iff j = i$

Proof of Prop 3: "$\exists a_i \in \sigma(A_i) \mid a_i \cap D(a_i) \neq \emptyset$"

- Let $a_1 \in \sigma(A_1)$
- If $a_1 \cap D(a_1) = \emptyset$, then $\forall j, \forall b \in \sigma(A_j)$, $a_1 \cap b = \emptyset$ $(*)$
- So, if $\forall a \in \sigma(A_1)$, we have $a \cap D(a) = \emptyset$, then $\sigma(A_1) \cup \sigma(A_2) \cup \ldots \cup \sigma(A_{n-1})$ is a set of non-intersecting curves.
- If so, ρ is cyclic. $(**)$

  $(**)$ uses a theorem of Lin:

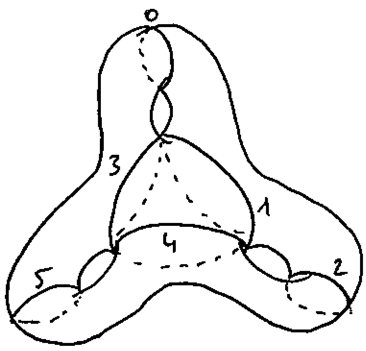
  $\boxed{\text{Any morphism from } B_n \text{ to } \sigma_{m<n} \text{ is cyclic.}}$

  and several ~~action~~ group actions of B_n on topological objects in Σ.

Proof of $(*)$

- $D^n(a_1) = a_1$ (quite hard)
- $\mathcal{A} = \{D^k(a_1),\ k \in \mathbb{Z}\}$ is a set of at most n curves.
- if $|\mathcal{A}| < n$, then $\forall j, \forall b \in \sigma(A_j)$ $a_1 \cap b = \emptyset$
- the case $|\mathcal{A}| = n$ with $a_1 \cap D(a_1) = \emptyset$ is <u>impossible</u>

$\langle D \rangle$ ↻

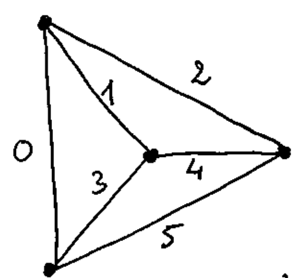


$j \leftrightarrow D^j(a_1)$

Surface Σ with set of curves $\mathcal{A}$ and a cyclic action on Σ that acts transitively on $\mathcal{A}$.

$\longleftrightarrow$ $\mathbb{Z}$ ↻

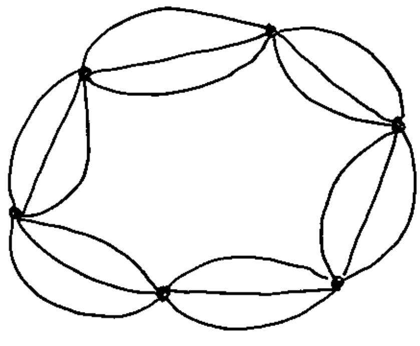
(correspondance)



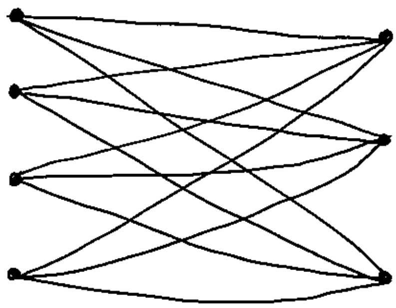
Graph with a $\mathbb{Z}$-action that is transitive on the edges

But such graph together with such a $\mathbb{Z}$-action is of one of the two following type:

type 1: (only one orbit of vertices)



type 2: with two orbits of vertices



These graphs have more than $\frac{n}{2}$ independent cycles, so the corresponding surface Σ has a genus $g > \frac{n}{2}$, but according to our hypotheses $g \leq \frac{n}{2}$.

So the case $|\mathcal{A}| = n$ is impossible. ∎