Anisotropic finite element methods with their applications

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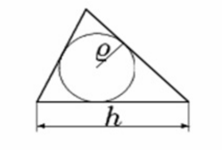
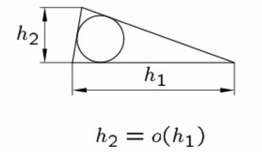
1 Backgrounds

Anisotropic FEM VS Isotropic FEM

- Anisotropic FEM (regularity assumption in Ciarlet [1978] or nondegenerate condition in Brenner, Scott [1994]):The length of the longest edge should be comparable with that of the diameter of the inscribed ball

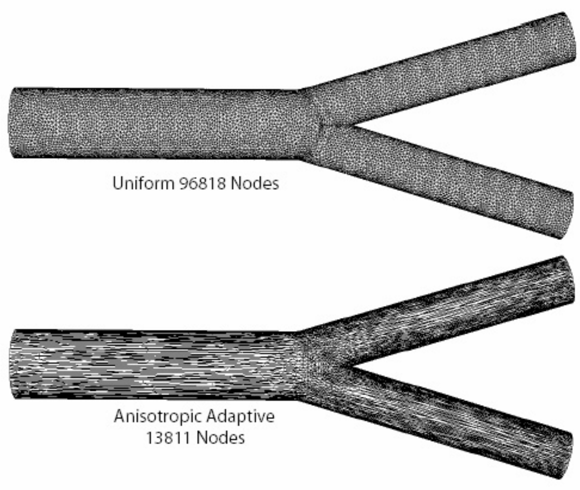


- Anisotropic (degenerate) FEM:



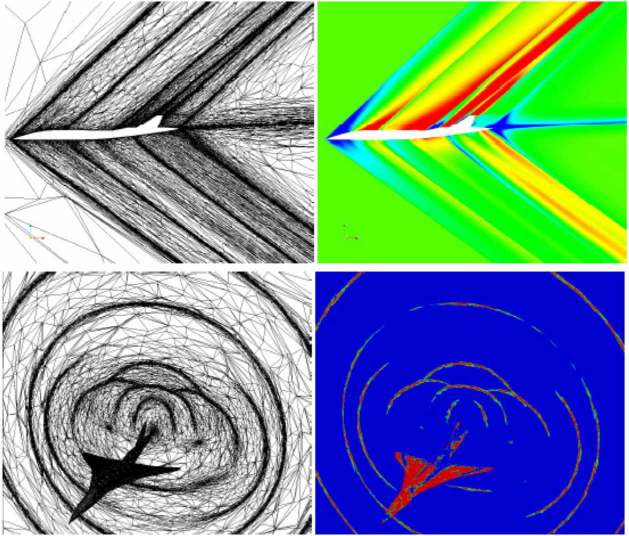
- Anisotropic finite element method behaves better in many practical problems!

Numerical Simulation of the flow in the blood vessels (Sahni, Mueller [05,06])



Anisotropic FEM can save the computational cost significantly with the same accuracy as the isotropic FEM

Anisotropic mesh for the inviscid flow around a supersonic jet (Bourgault et al [07])



The application of anisotropic FEM

- Many partial differential equations (PDEs) arising from science and engineering have a common feature that they have a small portion of the physical domain where small node separations are required to resolve large solution variations.
- Examples include problems having boundary layers, shock waves, ignition fronts, and/or sharp interfaces in fluid dynamics, the combustion and heat transfer theory, and groundwater hydrodynamics.
- Numerical solution of these PDEs using a uniform mesh may be formidable when the systems involve more than two spatial dimensions since the number of mesh nodes required can become very large. On the other hand, to improve efficiency and accuracy of numerical solution it is natural to put more mesh nodes in the region of large solution variation than the rest of the physical domain.
- The development of mathematical theory

2 Mathematical theory of anisotropic FEM

- Talk from Netwon's interpolation polynomial
- From the viewpoint of Girault- Raviart interpolation
- From the viewpoint of orthogonal expansions
- Anisotropic H^2 elements

Some works

- Mathematical studies of anisotropic meshes can be traced back to Synge [1957].
- Error estimates of the continuous linear finite element with the maximal angle condition: Feng [1975], Babuska, Aziz [1976], Gregory[1975], Barnhill, Gregory [1976a,1976b], Jamet [1976], Oganesyan, Rukhovets[1979].
- Until 90's in the last century, especially in recent years, much attention is paid to anisotropic FEM, Apel, Nicaise [92,99], Becker [95], Chen et al.[04], Duran [99], [ICM06] report......
- Most papers are on one special (or type of) finite element by studying its interpolation operator.
- We want to analyze it in a unified way through three new viewpoints

Talk from Netwon's interpolation polynomial

Lemma 1

Let $x_0 < x_1 < \cdots < x_m$, be a uniform partition, $d = x_{i+1} - x_i, 0 \le i \le m-1$. Suppose $f(x)$ is sufficiently smooth, then

$$f[x_0, \cdots, x_m] = \frac{1}{m!d^m} \int_{x_0}^{x_1} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \mathrm{d}t_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} f^{(m)}(t_m)\, \mathrm{d}t_m. \quad (1)$$

Remark Lemma 1 is similar to Hermite-Gennochi Theorem.

Theorem 2

For all $0 \le l \le m$, $f[x_0, \cdots, x_m]$ can be expressed by

$$f[x_0, \cdots, x_m] = \sum_{i=0}^{m-l} c_i f[x_i, \cdots, x_{i+l}], \quad (2)$$

where c_i $(0 \le i \le m-l)$ is only dependent to l and d.

- The interpolation polynomial $\mathrm{I}f(x)$ of $f(x)$ satisfying $\mathrm{I}f(x_i) = f(x_i), 0 \le i \le m$, can be expressed in the following form

$$\mathrm{I}f(x) = \sum_{i=0}^{m} f[x_0, \cdots, x_i] \prod_{j=0}^{i-1} (x - x_j), \tag{3}$$

where $p_i(x)$ $(0 \le i \le m)$ $\in P_m$ and $p_i(x_j) = \delta_{ij},\ 0 \le i, j \le m$.

- Denote the reference element $\hat{K} = [0,1]^2,\ d = 1/k,\ k$ is a positive integer, $\hat{x}_i = \hat{y}_i = id,\ i = 0, \cdots, k$. Suppose $\hat{u}(\hat{x}, \hat{y}) \in C(\hat{K})$, then bi-$k$-interpolation polynomial $\hat{\mathrm{I}}\hat{u}$ of $\hat{u}$ satisfying $\hat{\mathrm{I}}\hat{u}(\hat{x}_i, \hat{y}_j) = \hat{u}(\hat{x}_i, \hat{y}_j)(0 \le i, j \le k)$ has the following expression

$$\hat{\mathrm{I}}\hat{u} = \sum_{i=0}^{k} \sum_{j=0}^{k} \hat{u}(\hat{x}_i, \hat{y}_j) \hat{p}_i(\hat{x}) \hat{p}_j(\hat{y}),$$

where $\hat{p}_i(t) \in P_k(\hat{K}),\ \hat{p}_i(\hat{x}_l) = \hat{p}_i(\hat{y}_l) = \delta_{il}, 0 \le i, l \le k$. Obviously

$$\hat{\mathrm{I}}\hat{u} = \hat{u},\ \forall \hat{u} \in Q_k, \tag{4}$$

where Q_k is the polynomial space of the degree $\le k$ with respect to each variable.

We can prove that

$$\begin{aligned}
&\widehat{u}[\widehat{x}_0,\cdots,\widehat{x}_i;\widehat{y}_0,\cdots,\widehat{y}_r]\\
&=\frac{k^{\alpha_1+\alpha_2}}{\alpha_1!\alpha_2!}\sum_{j=0}^{i-\alpha_1}\sum_{s=0}^{r-\alpha_2}c_{js}\int_{\widehat{x}_j}^{\widehat{x}_{j+1}}\mathrm{d}t_1\int_{t_1}^{t_1+d}\cdots\mathrm{d}t_{\alpha_1-1}\int_{t_{\alpha_1-1}}^{t_{\alpha_1-1}+d}\\
&\left[\int_{\widehat{y}_s}^{\widehat{y}_{s+1}}\mathrm{d}s_1\int_{s_1}^{s_1+d}\cdots\mathrm{d}s_{\alpha_2-1}\int_{s_{\alpha_2-1}}^{s_{\alpha_2-1}+d}\frac{\partial^{\alpha_1+\alpha_2}\widehat{u}(\widehat{x},\widehat{y})}{\partial\widehat{x}^{\alpha_1}\partial\widehat{y}^{\alpha_2}}\,\mathrm{d}\widehat{y}\right]\mathrm{d}\widehat{x}\\
&\triangleq\widehat{L}(\widehat{D}^{\alpha}\widehat{u}).
\end{aligned}\tag{5}$$

Theorem 3

Suppose that $1 < p, q < \infty$, $W^{k+1,p}(\hat{K}) \hookrightarrow W^{m,q}(\hat{K})$, $W^{k+1,p}(\hat{K}) \hookrightarrow C^0(\hat{K})$, α *is an index,* $|\alpha| = m$, *then there exists a constant* $\hat{c} > 0$ *such that*

$$\|\widehat{D}^{\alpha}(\widehat{u} - \widehat{\mathrm{I}}\widehat{u})\|_{0,q,\widehat{K}} \leq \widehat{c}|\widehat{D}^{\alpha}\widehat{u}|_{k+1-m,p,\widehat{K}}. \tag{6}$$

- We can prove the above result hold for rectangular, cubic, triangular and tetrahedral Lagrange elements of arbitrary order!

Let K be a triangle (a tetrahedron) with the vertexes P_0, P_1, P_2 (P_0, P_1, P_2, P_3), $\underset{\sim}{v}_1, \underset{\sim}{v}_2$ $(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3)$ be the unit vectors along edges P_0P_1, P_0P_2 (P_0P_1, P_0P_2, P_0P_3) with $l_i = \|P_0P_i\|$, $\angle P_0$ be the maximum angle of the triangle K.
The affine mapping $F : \hat{K} \to K$ is

$$X = F(\widehat{X}) = B_K\widehat{X} + P_0, \tag{7}$$

where

$$B_K = B_{0K}\Lambda_K \tag{8}$$

$B_{0K} = (\underset{\sim}{v}_1, \underset{\sim}{v}_2),\ \Lambda = diag(l_1, l_2)$ for the triangular and
$B_{0K} = (\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3),\ \Lambda = diag(l_1, l_2, l_3)$ for the tetrahedron.

- Similar notations can work for parallelogram and parallelepiped elements.

Theorem 4

Under the same assumption as Theorem 3, then for rectangular (parallelogram), cubic (parallelepiped), triangular and tetrahedral Lagrange elements of arbitrary order, we have

$$|u - \mathrm{I}u|_{m,q,K} \leq C(detB_{0K})^{-m}(detB_K)^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{|\beta|=k+1-m} l^{\beta p} |D_l^{\beta} u|^p_{W_l^{m,p}(K)} \right)^{\frac{1}{p}}, \quad (9)$$

here the constant C does not depend on any geometrical conditions of K.

From the viewpoint of Girault- Raviart interpolation

- A special interpolation operator introduced by Girault and Raviart in [Finite element methods for Navier-Stokes equations, Theory and algorithms, Springer Series in Computational Mathematics, Springer, Berlin, 1986.](pp.100).
- Instead of the usual nodal Lagrange interpolant, Girault and Raviart introduce the following "vertex-edge-face" type interpolant, which is defined as

$$\begin{cases} \Pi_K^k u(a_i) = u(a_i), \;\; 1 \le i \le 3, \\ \text{if } \; k \ge 2 \;\; \int_l (\Pi_K^k u - u) p \, ds = 0, \;\; \forall p \in P_{k-2}(l), \;\; \forall \text{side } \; l \; \text{ of } \; K, \\ \text{if } \; k \ge 3 \;\; \int_K (\Pi_K^k u - u) p \, dxdy, \;\; \forall p \in P_{k-3}(K). \end{cases} \tag{2.2}$$

The global interpolant $\Pi_h^k : H^2(\Omega) \longrightarrow V_h$ is defined by $\Pi_h^k|_K = \Pi_K^k, \forall K \in J_h$.

Theorem 5

Assume that $u \in H^{k+1}(K)$, then we have

$$|u - \Pi_K^k u|_{1,K} \leq \frac{C}{|detB_{0K}|} h^k |u|_{k+1,K}, \tag{10}$$

where the constant C is independent of the geometric conditions of the triangle K.

- The above theorem is also hold for the interpolants proposed by Girault and Raviart for rectangular, cubic and tetrahedral Lagrange elements.

Proof. Without loss of generality, we assume that l_1, l_2 are the two edges of the maximal angle $\alpha_{M,K}$ and adopt the notations of Figure 1.

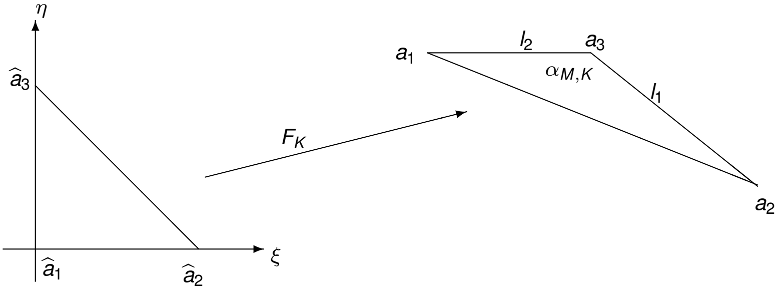


Figure 1

Let $\mathbf{s}_1 = (s_{11}, s_{12}), \mathbf{s}_2 = (s_{21}, s_{22})$ be the directions of the edges l_1 and l_2, respectively. Then it can be checked easily that

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{\sin \alpha_{M,K}} \begin{pmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial s_1} \\ \frac{\partial}{\partial s_2} \end{pmatrix}. \quad (2.4)$$

By an immediate computation, we have

$$\begin{aligned}|u-\Pi u|_{1,K}^2 &= \frac{1}{\sin\alpha_{M,K}^2}\left(\left\|\frac{\partial(u-\Pi u)}{\partial s_1}\right\|_{0,K}^2+\left\|\frac{\partial(u-\Pi u)}{\partial s_2}\right\|_{0,K}^2\right.\\ &\qquad\left.-2\cos\alpha_{M,K}\int_K\frac{\partial(u-\Pi u)}{\partial s_1}\frac{\partial(u-\Pi u)}{\partial s_2}dxdy\right)\\ &\le\frac{2}{\sin\alpha_{M,K}^2}\left(\left\|\frac{\partial(u-\Pi u)}{\partial s_1}\right\|_{0,K}^2+\left\|\frac{\partial(u-\Pi u)}{\partial s_2}\right\|_{0,K}^2\right).\end{aligned}\tag{2.5}$$

Set $V=\frac{\partial(u-\Pi u)}{\partial s_1}$, then by (2.2) there hold

$$\int_K Vp\,dxdy=0,\quad\forall p\in P_{k-2}(K),\ \text{ if }\ k\ge 2\tag{2.6}$$

and

$$\int_{l_1} Vp\,ds=0,\quad\forall p\in P_{k-1}(l_1).\tag{2.7}$$

Now we will prove on the reference element $\widehat{K}$ that

$$\|\widehat{V}\|_{0,\widehat{K}}\le C|\widehat{V}|_{k,\widehat{K}}.\tag{2.8}$$

Let us consider the interpolation operator $\widehat{I}: H^k(\widehat{K})\longrightarrow P_{k-1}(\widehat{K})$ defined by

$$\int_{\widehat{K}}\widehat{I}\widehat{v}p\,d\xi d\eta=\int_{\widehat{K}}\widehat{v}p\,d\xi d\eta,\quad\forall p\in P_{k-2}(\widehat{K}),\ \text{ if }\ k\ge 2\tag{2.9}$$

and

$$\int_{\widehat{l}_1} \widehat{I}\widehat{v}p\, d\widehat{s} = \int_{\widehat{l}_1} \widehat{v}p\, d\widehat{s}, \ \ \forall p \in P_{k-1}(\widehat{l}_1). \tag{2.10}$$

It can be checked that the above interpolation problem is well posed (the case $k = 1$ is trivial and we only need to consider $k \geq 2$). In fact, since (2.9) and (2.10) consists of $\frac{k(k+1)}{2}$ equations, which is just the dimension of $P_{k-1}(\widehat{K})$. Hence (2.9) and (2.10) is a square system of linear equations and it suffices to prove that its solution is unique. Thus, we assume that $q \in P_{k-1}(\widehat{K})$ satisfies

$$\int_{\widehat{K}} qp\, d\xi d\eta = 0, \ \ \forall p \in P_{k-2}(\widehat{K}) \tag{2.11}$$

and

$$\int_{\widehat{l}_1} qp\, d\widehat{s} = 0, \ \ \forall p \in P_{k-1}(\widehat{l}_1). \tag{2.12}$$

From (2.12) we can see that $q|_{\widehat{l}_1} \equiv 0$ and q can be expressed in term of barycentric coordinates as $q = \lambda_1 q_1$ with $q_1 \in P_{k-2}(\widehat{K})$, then taking $p = q_1$ in (2.11) we get that $q_1 \equiv 0$ and hence $q \equiv 0$.

Noticing that $\widehat{I\widehat{V}} = 0$, then (2.8) follows by an application of the Bramble-Hilbert lemma. By a scaling argument of (2.8) gives that

$$\left\| \frac{\partial(u - \Pi u)}{\partial s_1} \right\|_{0,K} \leq Ch^k \left| \frac{\partial u}{\partial s_1} \right|_{k,K}. \tag{2.13}$$

From the viewpoint of orthogonal expansions

Concerning the rectangular and cubic elements, it is known that the choices of the shape functions has much more freedom, thus there exists a variety of finite elements spaces. The above analysis is only covered the bi-k tensor-product elements.
On the reference element $\widehat{K} = \{-1 < \xi, \eta < 1\}$, we consider the following family shape functions:

$$\mathcal{Q}_m(k) = \sum_{(i,j)\in I_{k,m}} b_{i,j}\xi^i\eta^j, 1 \leq m \leq k, \tag{11}$$

where $I_{k,m}$ is a index, which satisfies that

$$\{(i,j)|0 \leq i,j \leq k, i+j \leq m+k\} \subset I_{k,m} \subset \{(i,j)|0 \leq i,j \leq k\}. \tag{12}$$

It includes the following three famous families of elements:

- i) Intermediate elements $\mathcal{Q}_1(k)$ (Babuska);
- ii) Uniform family $\mathcal{Q}_2(k), k \geq 2$, the special case is the bi-k tensor-product family $\mathcal{Q}_k(k)$;
- iii) Serendipity elements (Babuska), between P_k and $\mathcal{Q}_1(k)$.

We consider the Legendre polynomials defined on the interval $E = (-1, 1)$

$$L_n(t) = c_1(n)\frac{d^n(t^2-1)^n}{dt}, \quad c_1(n) = \frac{1}{2^n n!}, \quad n = 0, 1, 2, ... \qquad (13)$$

which is a series of orthogonal polynomials on E.
Furthermore, there holds

$$\|L_n\|_{0,E} = \sqrt{\frac{2}{2n+1}}, n = 0, 1, 2, ...$$

$L_n(t)$ has n vanish points on E, which is the so-called Gauss points. On the two end points of E, we have

$$L_n(\pm 1) = (\pm 1)^n.$$

We define

$$\phi_{n+1}(t) = \int_{-1}^{t} L_n(t)dt = c_1(n)\frac{d^{n-1}(t^2-1)^n}{dt}, \quad n = 0, 1, 2, ... \qquad (14)$$

which is the so-called Lobatto polynomials.
When $n \geq 2$ we have $\phi_n(\pm 1) = 0$. Lobatto polynomials have the following quasi-orthogonal relationship:

$$\int_E \phi_m(t)\phi_n(t)dt = \begin{cases} \neq 0, \text{if} \quad m-n = 0, \pm 2; \\ 0, \text{else} \quad m, n. \end{cases} \qquad (15)$$

If $u \in W^{1,1}(E)$, we can do an orthogonal expansion by Legendre polynomials for $\frac{du}{dt}$,

$$\frac{du}{dt} = \sum_{j=1}^{\infty} b_j L_{j-1}(t), \tag{16}$$

$$b_j = (j - \frac{1}{2}) \int_E \frac{du}{dt} L_{j-1}(t) dt, \quad j = 1, 2, ...$$

Integrating the both hand sides of (25) on (-1,t), we get

$$u(t) = \sum_{j=0}^{\infty} b_j \phi_j(t).$$

Now we can get a n order polynomial expansion of u, which is denoted by

$$u_n = \sum_{j=0}^{n} b_j \phi_j(t).$$

In order to determine the value of b_0,, we assume u is C^0 continuous on the endpoints $t = \pm 1$, then

$$u(1) = b_0 + b_1, u(-1) = b_0 - b_1,$$

from which we can derive

$$b_0 = \frac{u(1) + u(-1)}{2}, \quad b_1 = \frac{1}{2} \int_E \frac{du}{dt} L_0(t) dt = \frac{u(1) - u(-1)}{2}.$$

We construct the interpolation function on the reference square as

$$\widehat{\Pi}\widehat{u} = \sum_{(i,j)\in I_{k,m}} b_{i,j}\phi_i(\xi)\phi_j(\eta) \in \mathcal{Q}_m(k). \tag{17}$$

where

$$\begin{aligned}
b_{0,0} &= \frac{\widehat{u}_1 + \widehat{u}_2 + \widehat{u}_3 + \widehat{u}_4}{4},\\
b_{0,j} &= \frac{2j-1}{4}\int_{-1}^{1}\left(\frac{\partial\widehat{u}(1,\eta)}{\partial\eta} + \frac{\partial\widehat{u}(-1,\eta)}{\partial\eta}\right)L_{j-1}(\eta)d\eta, j \geq 1,\\
b_{i,0} &= \frac{2i-1}{4}\int_{-1}^{1}\left(\frac{\partial\widehat{u}(\xi,1)}{\partial\xi} + \frac{\partial\widehat{u}(\xi,-1)}{\partial\xi}\right)L_{i-1}(\xi)d\xi, i \geq 1,\\
b_{i,j} &= \frac{(2i-1)(2j-1)}{4}\int_{\widehat{K}}\frac{\partial^2\widehat{u}(\xi,\eta)}{\partial\xi\partial\eta}L_{i-1}(\xi)L_{j-1}(\eta)d\xi d\eta, i,j \geq 1.
\end{aligned} \tag{18}$$

Theorem 6

Assume $1 \le p, q \le \infty, \alpha = (\alpha_1, \alpha_2) \in I_{k,m}$ *is an index,* $\widehat{u} \in C^0(\widehat{K})$ *satisfies that* $\widehat{D}^\alpha \widehat{u} \in W^{k+1-|\alpha|,p}(\widehat{K})$*, if we take p an integer l such that* $0 \le |\alpha| \le l \le k+1$,

$$\begin{cases} p = \infty, & \text{for } |\alpha| = 0, l = 0, \\ p > 2, & \text{for } |\alpha| = 0, l = 1, \\ 0 < |\alpha| < l, & \text{for } \alpha_1 = 0 \text{ or } \alpha_2 = 0, \end{cases} \tag{19}$$

$W^{l-|\alpha|,p}(\widehat{K}) \hookrightarrow L^q(\widehat{K})$*, If* $n \le k+1$*, we have*

$$|u - \Pi u|_{n,q,K} \le C(detB_K)^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{|\beta|=l-n} h^{\beta p} |D^\beta u|^p_{l-n,p,K} \right)^{\frac{1}{p}}. \tag{20}$$

Else if $n > k+1$*, we have*

$$|u - \Pi u|_{n,q,K} \le C(detB_K)^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{\beta \in D^\alpha I_{k,m}} h^{\beta p} |D^\beta u|^p_{l-n,p,K} \right)^{\frac{1}{p}}. \tag{21}$$

Anisotropic H^2 elements

- Most references treated the anisotropic H^1 finite elements. The analysis of anisotropic H^2 interpolation error estimates is missing.
- We discuss the method to construct arbitrary order H^2 (C^1 continuous) Hermite elements.
- Anisotropic interpolations of them are obtained by the orthogonal expansion technique.

We define aseries of polynomials by integral the Lobatto polynomials on (-1,t),

$$\psi_0 = 1, \psi_1 = t, \psi_2 = (t^2 - 1)/2, \psi_3 = (t^3 - 3t)/6, ...$$

which can be written as

$$\psi_{n+2} = \int_{-1}^{t} \phi_{n+1}(t)dt = c_1(n)\frac{d^{n-2}(t^2-1)}{dt}, n = 2, 3, 4... \tag{22}$$

moreover, there stand

$$\psi_n(\pm 1) = 0, \quad n \geq 4, \tag{23}$$

and

$$\int_{-1}^{1} \psi_n(t)qdt = 0, \quad \forall q \in P_{n-5}(E), \quad n \geq 5. \tag{24}$$

If $u \in W^{2,1}(E)$, we can do a orthogonal expansion by Legendre polynomials for $\frac{d^2u}{dt^2}$, which yields

$$\frac{d^2u}{dt^2} = \sum_{j=0}^{\infty} b_{j+2}L_j(t), \tag{25}$$

where

$$b_{j+2} = (j - \frac{1}{2})\int_E \frac{d^2u}{dt^2}L_j(t)dt, \quad j = 0, 1, 2, ...$$

Integrating the both hand sides of (25) on (-1,t), we get

$$\frac{du}{dt} = b_1 + \sum_{j=0}^{\infty} b_{j+2}\phi_{j+1}(t),$$

Integrating the above again, we have

$$u = b_0 + b_1 t + \sum_{j=0}^{\infty} b_{j+2}\psi_{j+2}(t).$$

Now we can get a n order polynomial expansion of u, which is denoted by

$$u_n = \sum_{j=0}^{n} b_j \psi_j(t).$$

In order to determine the value of b_0, b_1, we assume u is C^1 continuous on the endpoints $t = \pm 1$, then

$$u_t(1) = b_1 + b_2, u_t(-1) = b_1 - b_2,$$
$$u(1) = b_0 + b_1 - \frac{1}{3}b_3, u(-1) = b_0 - b_1 + \frac{1}{3}b_3,$$

together with

$$b_2 = \frac{1}{2}\int_{-1}^{1} \frac{d^2u}{dt^2}dt = \frac{1}{2}(u_t(1) - u_t(-1)),$$
$$b_3 = \frac{3}{2}\int_{-1}^{1} \frac{d^2u}{dt^2}tdt = \frac{3}{2}(u_t(1) + u_t(-1) - u(1) + u(-1))$$

we can derive

$$b_0 = \frac{1}{2}(u(1) + u(-1)), b_1 = \frac{1}{2}(u_t(1) + u_t(-1)).$$

Now we define the rectangular elements. On the reference element $\widehat{K} = \{-1 < \xi, \eta < 1\}$, we consider the following shape functions

$$\mathcal{Q}_m(k) = \sum_{(i,j)\in I_{k,m}} b_{i,j}\xi^i\eta^j, 3 \leq m \leq k, \tag{26}$$

where $I_{k,m}$ is an index set satisfied

$$\{(i,j)|0 \leq i,j \leq k, i+j \leq m+k, 3 \leq m \leq k\} \subset I_{k,m} \subset \{(i,j)|0 \leq i,j \leq k\}. \tag{27}$$

We can get the interpolation on the reference element which is expressed as

$$\widehat{\Pi}\widehat{u} = \sum_{(i,j)\in I_{k,m}} b_{i,j}\psi_i(\xi)\psi_j(\eta) \in \mathcal{Q}_m(k), \tag{28}$$

where

$$
\begin{aligned}
b_{0,0} &= \frac{\widehat{u}_1+\widehat{u}_2+\widehat{u}_3+\widehat{u}_4}{4}, \; b_{1,0} = \frac{\widehat{u}_{1\xi}+\widehat{u}_{2\xi}+\widehat{u}_{3\xi}+\widehat{u}_{4\xi}}{4},\\
b_{0,1} &= \frac{\widehat{u}_{1\eta}+\widehat{u}_{2\eta}+\widehat{u}_{3\eta}+\widehat{u}_{4\eta}}{4}, \; b_{1,1} = \frac{\widehat{u}_{1\xi\eta}+\widehat{u}_{2\xi\eta}+\widehat{u}_{3\xi\eta}+\widehat{u}_{4\xi\eta}}{4},\\
b_{i,0} &= \frac{2i-3}{4}\int_{-1}^{1}\left(\frac{\partial^2\widehat{u}(\xi,1)}{\partial\xi^2}+\frac{\partial^2\widehat{u}(\xi,-1)}{\partial\xi^2}\right)L_{i-2}(\xi)d\xi, \; i\geq 2,\\
b_{0,j} &= \frac{2j-3}{4}\int_{-1}^{1}\left(\frac{\partial^2\widehat{u}(1,\eta)}{\partial\eta^2}+\frac{\partial^2\widehat{u}(-1,\eta)}{\partial\eta^2}\right)L_{j-2}(\eta)d\eta, \; j\geq 2,\\
b_{i,1} &= \frac{2i-3}{4}\int_{-1}^{1}\left(\frac{\partial^3\widehat{u}(\xi,1)}{\partial\xi^2\partial\eta}+\frac{\partial^3\widehat{u}(\xi,-1)}{\partial\xi^2\partial\eta}\right)L_{i-2}(\xi)d\xi, \; i\geq 2,\\
b_{1,j} &= \frac{2j-3}{4}\int_{-1}^{1}\left(\frac{\partial^3\widehat{u}(1,\eta)}{\partial\xi\partial\eta^2}+\frac{\partial^3\widehat{u}(-1,\eta)}{\partial\xi\partial\eta^2}\right)L_{j-2}(\eta)d\eta, \; j\geq 2,\\
b_{i,j} &= \frac{(2i-3)(2j-3)}{4}\int_{\widehat{K}}\frac{\partial^4\widehat{u}(\xi,\eta)}{\partial\xi^2\partial\eta^2}L_{i-2}(\xi)L_{j-2}(\eta)d\xi d\eta, \; i,j\geq 2,
\end{aligned}
\tag{29}
$$

here $\widehat{u}_i, \widehat{u}_{i\xi}, \widehat{u}_{i\eta}, \widehat{u}_{i\xi\eta}, i=1,2,3,4$ denote the function values and corresponding derivative values of $\widehat{u}$ on the four vertexes.

Theorem 7

Assume $1 \le p, q \le \infty, \alpha = (\alpha_1, \alpha_2) \in I_{k,m}$ *is a multi-index,* $\widehat{u} \in C^2(\widehat{K})$ *satisfies* $\widehat{D}^\alpha \widehat{u} \in W^{k+1-|\alpha|,p}(\widehat{K})$*, if p and a positive integer l satisfy:* $0 \le \max\{|\alpha|, 2\} \le l \le k+1$,

$$\begin{cases} p = \infty, & \text{if } |\alpha| = 0, l = 2, \\ p > 2, & \text{if } |\alpha| = 0, l = 3, \\ 0 < |\alpha| < l, & \text{if } \alpha_1 = 0, 1 or \alpha_2 = 0, 1, \end{cases} \tag{30}$$

$W^{l-|\alpha|,p}(\widehat{K}) \hookrightarrow L^q(\widehat{K})$*, then there exists a constant* $C > 0$ *such that*

$$\|\widehat{D}^\alpha(\widehat{u} - \widehat{\Pi}\widehat{u})\|_{0,q,\widehat{K}} \le C|\widehat{D}^\alpha \widehat{u}|_{l-|\alpha|,p,\widehat{K}}. \tag{31}$$

If $k+1 < |\alpha|$*, we have*

$$\|\widehat{D}^\alpha(\widehat{u} - \widehat{\Pi}\widehat{u})\|_{0,q,\widehat{K}} \le C\left(\sum_{\beta \in D^\alpha I_{k,m}} \|\widehat{D}^{\alpha+\beta}\widehat{u}\|^p_{0,p,\widehat{K}}\right)^{\frac{1}{p}}, \tag{32}$$

where $D^\alpha I_{k,m} := \{(i-\alpha_1, j-\alpha_2) | (i,j) \in) I_{k,m}\}$

Theorem 8

Under the same assumptions as that of theorem 7, if $n \leq k+1$, then we have

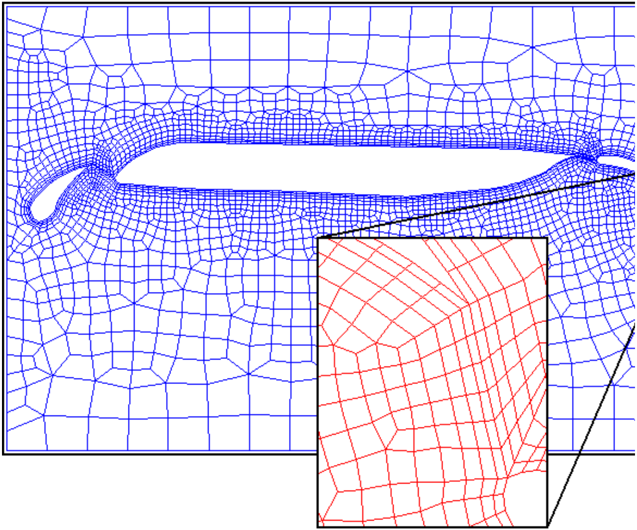
$$|u - \Pi_K u|_{n,q,K} \leq C(detB_K)^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{|\beta|=l-n} h_K^{\beta p} |D^\beta u|^p_{l-n,p,K} \right)^{\frac{1}{p}}. \quad (33)$$

else if $n > k+1$, we have

$$|u - \Pi_K u|_{n,q,K} \leq C(detB_K)^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{\beta \in D^\alpha I_{k,m}} h_K^{\beta p} |D^\beta u|^p_{l-n,p,K} \right)^{\frac{1}{p}}. \quad (34)$$

3 Degenerate quadrilateral elements

Quadrilateral finite elements, particularly low order quadrilateral elements, are widely used in engineering computations due to their flexibility and simplicity.



- It is known that Q_1 quadrilateral finite element is the mostly used quadrilateral element, in order to obtain the optimal interpolation error of it, many mesh conditions have been introduced in the references, let us give a review of them.
- The first interpolation error estimate for the Lagrange interpolation operator $\mathcal{Q}$ is Ciarlet and Raviart [1972], where the regular quadrilateral is defined as

$$h_K/\bar{h}_K \leq \mu_1 \tag{1.1}$$

$$|\cos\theta_K| \leq \mu_2 < 1 \tag{1.2}$$

  for all angle θ_K of quadrilateral K, here h_K, $\bar{h}_K$ is the length of the diameter and the shortest side of K, respectively.
- Under the above so-called “nondegenerate" condition, we have

$$|u - \mathcal{Q}u|_{1,K} \leq Ch_K|u|_{2,K}. \tag{1.3}$$

- For triangular elements, the constant C in the estimate (1.3) depends only on the maximal angle of the element.
- Question: what is the constant depends on for the quadrilateral elements?
- For quadrilateral elements, the mesh condition is quite different from the triangular case since its geometric condition is very complex.
- Jamet [SIAM 1977] shows that the maximal angle condition (1.2) is not necessary. He considered a quadrilateral can degenerate into a regular triangle
- Zenisek, Vanmaele [Numer. Math.1995,1996], Apel [Computing, 1998] show that condition (1.1) is not necessary.
- Acosta, Duran, [SIAM 2000] and [Numer. Math. 2006], RDP condition, which reads as

Definition 2.1. *A quadrilateral or a triangle verifies the maximal angle condition with constant $\psi < \pi$, or shortly MAC(ψ), if the interior angles of K are less than or equal to ψ*

Definition 2.2. *[Acosta, Duran, 2000] We say that K satisfies the regular decomposition property with constants N $\in$ R and $0 < \psi < \pi$, or shortly RDP(N, ψ), if we can divide K into two triangles along one of its diagonals, which will always be called d_1, the other be d_2 in such a way that $|d_2|/|d_1| \leq N$ and both triangles satisfy MAC(ψ).*

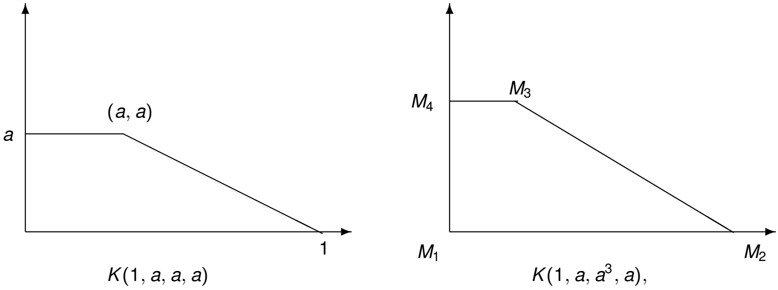
- The authors assert that this condition is necessary and state it as an open problem in the conclusion of their paper.
- Under RDP condition, Acosta, Duran [2006] proved the optimal interpolation error in $W^{1,p}$ norm with $1 \leq p < 3$

- Is the MAC condition necessary for Q_1 quadrilateral elements?
- Our motivation: if we divide the quadrilateral into two triangles by the longest diagonal, when the two triangles have the comparable area, we should impose the maximal angle condition for both triangles, otherwise, we may only need to impose the maximal angle condition for the big triangle T_1, and because the error on the small triangle T_3 contributes little to the interpolation error on the global quadrilateral.

Consider the case $K(1, a, a, a)$ and take $u = x^2$. Straightforward computations show that

$$\left\| \frac{\partial(u - \mathcal{Q}u)}{\partial y} \right\|_{0,K}^2 \geq Ca\ln(a^{-1}) \text{ and } |u|_{2,K}^2 \leq Ca.$$

Then the constant on the right hand of (1.3) can not be bounded when a approaches zero.



Figure

If one consider the case $K(1, a, a^3, a)$ (the right side of the above Figure), we have

$$\left\|\frac{\partial(u-\mathcal{Q}u)}{\partial y}\right\|_{0,K}^2 \le Ca^5\ln(a^{-1}), \quad |u|_{2,K}^2 \ge Ca. \tag{2.4}$$

However, in this case the error constant

$$\frac{\|\frac{\partial(u-\mathcal{Q}u)}{\partial y}\|_{0,K}^2}{|u|_{2,K}^2} \le Ca^4\ln(a^{-1}) \tag{2.5}$$

can be bounded with a constant independent of a.

- What has happened to the quadrilateral during this minor change?
- One reasonable interpretation is that the ratio $\frac{S_{\triangle M_2M_3M_4}}{S_{\triangle M_1M_2M_4}}$ of $K(1, a, a^3, a)$ is much smaller than that of $K(1, a, a, a)$, this can further relax the maximal angle condition of $\triangle M_2M_3M_4$ because the error on $\triangle M_2M_3M_4$ contributes less compared to that on $\triangle M_2M_1M_4$.

Definition 2.3. *We say that K satisfies the generalized regular decomposition property with constant* $N \in \mathcal{R}$ *and* $0 < \psi < \pi$*, or shortly GRDP(*N, ψ*), if we can divide K into two triangles along one of its diagonals, which will always be called* d_1*, in such a way that the big triangle satisfies MAC(*ψ*) and that*

$$\frac{h_K}{d_1 \sin\alpha}\left(\frac{|T_3|}{|T_1|}\ln\frac{|T_1|}{|T_3|}\right)^{\frac{1}{2}} \leq N, \tag{2.6}$$

where the big triangle will always be called T_1*, the other be* T_3*,* h_K *denotes the diameter of the quadrilateral K and* α *the maximal angle of* T_3*.*

- Let $a_3 = d_2 \cap T_3$ and $a_1 = d_2 \cap T_1$ denote the two parts of the diagonal d_2 divided by the diagonal d_1, thus the GRDP condition can be easily checked in practice computations, particularly, if we choose the longest diagonal for d_1, the condition becomes $\frac{1}{\sin\alpha}\left(\frac{|a_3|}{|a_1|}\ln\frac{|a_1|}{|a_3|}\right)^{\frac{1}{2}} \leq N$.
- It is easy to see that if a quadrilateral K satisfies the RDP condition, then it also satisfies the GRDP condition. However, the converse is not true, as shown by the example $K(1, a, a^s, a)$ with $s > 2$.
- Let us show that the condition in GRDP that the big triangle satisfies the maximal angle condition is necessary. Indeed consider the family of quadrilaterals Q_α of vertices $M_1 = (-1+\cos\alpha, -\sin\alpha)$, $M_2 = (1, 0)$, $M_3 = (1-\cos\alpha, \sin\alpha)$ and $M_4 = (-1, 0)$, with the parameter $\alpha \in (\frac{\pi}{2}, \pi)$. If we consider $u(x, y) = x^2$, for all $\alpha \in (\pi - \beta_0, \pi)$, the ratio

$$\frac{|u - \mathcal{Q}u|_{1,Q_\alpha}}{|u|_{2,Q_\alpha}} \geq \frac{C}{4\sin\alpha}$$

and hence goes to infinity as α tends to π.

Let Π be the conforming P_1 Lagrange interpolation operator on the big triangle T_1, then we have

$$|u - \mathcal{Q}u|_{1,K} \leq |\Pi u - \mathcal{Q}u|_{1,K} + |u - \Pi u|_{1,K}.$$

Because

$$(\Pi u - \mathcal{Q}u)(x) = (\Pi u - u)(M_3)\phi_3(x),$$

then we have

$$|u - \mathcal{Q}u|_{1,K} \leq |(\Pi u - u)(M_3)||\phi_3|_{1,K} + |u - \Pi u|_{1,K}.$$

In the subsequent analysis, we just adopt the notations as the following figure

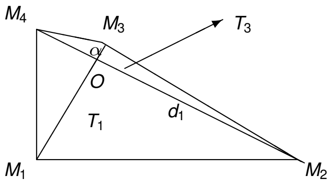


Figure 2. A general convex quadrilateral K

Lemma 3.1. *Let θ be the angle of the two diagonals M_1M_3 (denoted by d_2) and M_2M_4 (denoted by d_1) and let O be the point at which they intersect. Let $a_i = |OM_i|$ with $a_i > 0$ for $i = 1, 2, 4$ and $a_3 \geq 0$. Let α, s be the maximal angle and be the shortest edge of the triangle T_3, respectively, we have*

$$\int_{\widehat{K}} \frac{1}{|J|} d\xi d\eta < \frac{4}{|d_1||s| \sin\alpha} \frac{|T_3|}{|T_1|} \left(2 + \ln \frac{|T_1|}{|T_3|}\right). \tag{3.4}$$

Lemma 3.2. *Let K be a general convex quadrilateral with the same hypothesis as Lemma 3.1, we have*

$$\left|\phi_3\right|_{1,K} \leq \frac{8h_K}{(|d_1||s| \sin\alpha)^{\frac{1}{2}}} \left(\frac{|T_3|}{|T_1|} \left(2 + \ln \frac{|T_1|}{|T_3|}\right)\right)^{\frac{1}{2}}. \tag{3.10}$$

Note that Lemma 3.2. gives a sharp estimate of the term $|\phi_3|_{1,K}$ up to a generic constant. In fact, one can just consider the example of the quadrilateral $K(1, b, a, b)$ under the assumption $0 < a, b \ll 1$. Some immediate calculations yield

$$\left|\phi_3\right|_{1,K} \geq \left\|\frac{\partial \phi_3}{\partial y}\right\|_{0,K} \geq C\frac{1}{\sqrt{b(1-a)}}\left(\ln\frac{1}{a}\right)^{\frac{1}{2}}$$
$$\geq C\frac{h_K}{(|d_1||s|\sin\alpha)^{\frac{1}{2}}}\left(\frac{|T_3|}{|T_1|}\left(2+\ln\frac{|T_1|}{|T_3|}\right)\right)^{\frac{1}{2}}$$

since $|s| = a, \sin\alpha > b$ and $\frac{|T_3|}{|T_1|} = a$.

Lemma 3.3. Let K be a general convex quadrilateral, then we have

$$||(u - \Pi u)(M_3)| \leq \left(\frac{4|s|}{|d_1|\sin\alpha}\right)^{\frac{1}{2}} \left\{|u - \Pi u|_{1,T_3} + h_K|u|_{2,T_3}\right\}.$$

- The result of lemma 3.3 gives a sharp estimate up to a generic constant. Consider the example of the quadrilateral $K(1, b, a, b)$ under the assumption $0 < a, b \ll 1$ and the function $u(x, y) = x^2$. We then see that $\Pi u = x$ and therefore

$$|(\Pi u - u)(M_3)| = a(1 - a) \geq C\left(\frac{|s|}{|d_1|\sin\alpha}\right)^{\frac{1}{2}} \left\{|u - \Pi u|_{1,T_3} + h_K|u|_{2,T_3}\right\}$$

since $|s| = a, \sin\alpha > b$ and $|u - \Pi u|_{1,T_3} + h_K|u|_{2,T_3} \leq \sqrt{ab}$.

Lemma 3.4. *Let K be a general convex quadrilateral and Π be the linear Lagrange interpolation operator defined on T_1, then*

$$|u - \Pi u|_{1,K} \leq \frac{4}{\sin\gamma}\left(1 + \frac{2}{\pi}\right)\left(\frac{2|K|}{|T_1|}\right)^{\frac{1}{2}} h_K |u|_{2,K},$$

where γ is the maximal angle of T_1.

- This result is also sharp in view of the same above example.

Theorem 3.5. *Let K be a convex quadrilateral satisfies GRDP(N, ψ), then we have*

$$|u - \mathcal{Q}u|_{1,K} \leq Ch_K |u|_{2,K}, \tag{3.18}$$

with C only depending on N and ψ.
In fact, we have proved that

$$C \leq \left\{ \frac{32h_K}{|d_1|\sin\alpha} \left(\frac{|T_3|}{|T_1|} \left(2 + \ln \frac{|T_1|}{|T_3|} \right) \right)^{\frac{1}{2}} + 1 \right\} \frac{4}{\sin\gamma} \left(1 + \frac{2}{\pi} \left(\frac{2|K|}{|T_1|} \right)^{\frac{1}{2}} \right)$$

Let us denote by $C(K,p)$ a positive constant such that

$$|\phi_3|_{1,p,K} \leq h_K C(K,p).$$

Any method that furnishes a computable value of $C(K,p)$ will drive to a sufficient condition for interpolation error estimates in $W^{1,p}$. We develop a generic approach that will be applied for different values of p.

Definition 4.1. We say that K satisfies the generalized regular decomposition property with constants $N \in \mathbb{R}_+$, $0 < \psi < \pi$ and $p \in [1, \infty)$, or shortly GRDP(N, ψ, p), if we can divide K into two triangles along one of its diagonals, always called d_1, in such a way that the big triangle satisfies MAC(ψ) and that

$$\frac{h_K}{(2-p)^{\frac{1}{p}} |d_1| \sin\alpha} \left(\frac{|T_3|}{|T_1|}\right)^{1-\frac{1}{p}} \le N, \quad \text{for } p \in [1,2),$$

$$\frac{h_K}{d_1 \sin\alpha} \left(\frac{|T_3|}{|T_1|} \ln \frac{|T_1|}{|T_3|}\right)^{\frac{1}{2}} \le N, \quad \text{for } p = 2$$

$$\frac{h_K}{(p-2)^{\frac{1}{p}} (3-p)^{\frac{1}{p}} |d_1| \sin\alpha} \left(\frac{|T_3|}{|T_1|}\right)^{\frac{1}{p}} \le N, \quad \text{for } p \in (2,3),$$

$$\frac{h_K}{|d_1| \sin\alpha} \max\left\{1, \frac{|T_3|}{|T_4|}\right\}^{1-\frac{3}{p}+\frac{1}{2p}} \left(\frac{|T_3|}{|T_1|}\right)^{\frac{1}{p}} \le N, \quad \text{for } p \in [3, \frac{7}{2}],$$

$$\frac{h_K}{|d_1| \sin\alpha} \max\left\{1, \frac{|T_3|}{|T_4|}\right\}^{\frac{1}{p}} \left(\frac{|T_3|}{|T_1|}\right)^{\frac{1}{p}} \le N, \quad \text{for } p \in (\frac{7}{2}, 4],$$

$$\frac{h_K}{|d_1| \sin\alpha} \max\left\{1, \frac{|T_3|}{|T_4|}\right\}^{1-\frac{3}{p}} \left(\frac{|T_3|}{|T_1|}\right)^{\frac{1}{p}} \le N, \quad \text{for } p > 4,$$

- For H^1 norm, our condition is weaker than RDP condition proposed by Acosta, Duran [SINUM 2000]
- For $W^{1,p}$ norm with $1 \leq p \leq 3$, our condition is weaker than RDP condition proposed by Acosta, Monzon [NM 2006]
- For $W^{1,p}$ norm with $p > 3$, our condition is weaker than the double angle condition (DAC) proposed by Acosta, Monzon [2006], which needs all the interior angles ω of K verify $0 < \psi_m \leq \omega \leq \psi_M < \pi$. In fact, the DAC$(\psi_m, \psi_M)$ condition is a quite strong geometric condition and the following elementary implications hold:

$$\text{DAC}(\psi_m, \psi_M) \Longrightarrow \text{MAC}(\psi_M) \Longrightarrow \text{RDP}(N, \psi_M, p) \Longrightarrow \text{GRDP}(N, \psi_M)$$

- Our GRDP condition is stated in a continuous way with respect to p

4 Application to problems with singularities

We consider the following simple model problem: Find $u \in H_0^1(\Omega)$, such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u|_\Gamma = 0, & \text{on } \Gamma. \end{cases} \tag{35}$$

$$\|u\|_{k,l,\beta} = \Bigg(\|u\|^2_{k,\Omega_0} + \sum_{(i,j)\in\mathfrak{I}} \|u\|^2_{k,l,\beta,\mathfrak{U}_{ij}} + \sum_{m\in\mathcal{M}} \|u\|^2_{k,l,\beta,\widetilde{\mathcal{O}}_m} + \sum_{(i,j)\in\mathfrak{I}} \sum_{m\in\mathcal{M}} \|u\|^2_{k,l,\beta,\beta,V_{m,ij}} \Bigg)^{\frac{1}{2}}$$

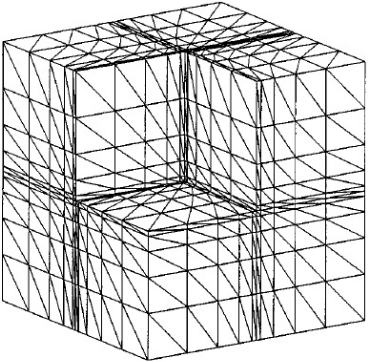
Theorem 9

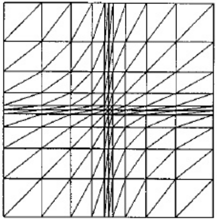
Given $\overline{\beta} \le \beta < \frac{1}{2}$, if $f \in H_\beta^{k,0}(\Omega)$, then problem (35) exists a unique solution $u \in H_\beta^{k+2,2}(\Omega)$ and

$$\|u\|_{k+2,2,\beta} \le C\|f\|_{k,0,\beta}. \tag{36}$$

We take a parameter $\mu \in (0,1]$ to describe the graded meshes. For any element $K \in \mathcal{J}_h$, such that

$$\begin{cases} l_1 \sim l_2 \sim l_3 \sim h^{\frac{1}{\mu}}, \quad K \in \mathcal{J}_{1h}, \\ l_1 \sim l_2 \sim h^{\frac{1}{\mu}}, l_3 \sim h, \quad K \in \mathcal{J}_{2h} \bigcap \mathcal{J}_{\mathfrak{U}_{ij},h}, \\ l_1 \sim l_2 \sim h r_{ij,K}^{1-\mu}, l_3 \sim h, \quad K \in \mathcal{J}_{\mathfrak{U}_{ij},h} \setminus \mathcal{J}_{2h}, \\ l_1 \sim l_2 \sim h^{\frac{1}{\mu}}, l_3 \sim h\rho_{m,K}^{1-\mu}, \quad K \in (\mathcal{J}_{2h} \setminus \mathcal{J}_{1h}) \bigcap \mathcal{J}_{V_{m,ij},h}, \\ l_1 \sim l_2 \sim h r_{ij,K}^{1-\mu}, l_3 \sim h\rho_{m,K}^{1-\mu}, \quad K \in \mathcal{J}_{V_{m,ij},h} \setminus \mathcal{J}_{2h}, \\ l_1 \sim l_2 \sim l_3 \sim h\rho_{m,K}^{1-\mu}, \quad K \in \mathcal{J}_{\tilde{\mathcal{O}}_m,h} \setminus \mathcal{J}_{2h}, \\ l_1 \sim l_2 \sim l_3 \sim h, \quad K \in \mathcal{J}_{\Omega_0,h}. \end{cases} \tag{37}$$





A simple case

We consider the anisotropic finite element approximation of the problem (35):

$$\begin{cases} \text{Find} \quad u_h^{k+1} \in V_h^{k+1}, \text{ such that} \\ a(u_h^{k+1}, v_h^{k+1}) = f(v_h^{k+1}), \forall\, v_h^{k+1} \in V_h^{k+1}. \end{cases} \tag{38}$$

Theorem 10

Assume u, u_h^{k+1} are the solutions of (35) and (38), respectively, the meshes satisfy the condition (37), $\mu \leq \frac{1-\beta}{k+1}$, then under the assumptions of Theorem 9, we have

$$|u - u_h^{k+1}|_{1,\Omega} \leq Ch^{k+1}\|f\|_{k,2,\beta,\Omega}. \tag{39}$$

5 Application to singularly perturbed problems

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Thank you for your attention!