Finite volume schemes for non-coercive elliptic problems with Neumann boundary conditions

Claire CHAINAIS-HILLAIRET

Laboratoire de Mathématiques
Université Blaise Pascal, Clermont-Ferrand

Joint work with Jérôme DRONIOU, Montpellier II

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Outline of the talk

1. Introduction to the problem
2. Presentation of the finite volume schemes
3. Existence and uniqueness of a solution
4. Results of convergence
5. Numerical experiments, comparison of the different schemes

Introduction to the problem

Problem under study

$$(\mathcal{P}_0)\ \begin{cases} \mathrm{div}(\mathbf{J}) = g, \text{ with } \mathbf{J} = -\nabla u + \mathbf{V}u, \text{ in } \Omega \\ \mathbf{J}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \end{cases}$$

Hypotheses

- Ω open bounded polygonal connected domain of $\mathbb{R}^d$,
- $g \in L^2(\Omega)$,
- $\mathbf{V} \in L^p(\Omega)^d \quad (2 < p < +\infty \text{ if } p = 2,\ p = d \text{ if } d \geq 3)$.

Weak solution

$$\begin{cases} \bar{u} \in H^1(\Omega)\,, \\ \forall \varphi \in H^1(\Omega)\,,\ \int_\Omega \nabla\bar{u}\cdot\nabla\varphi - \int_\Omega \bar{u}\mathbf{V}\cdot\nabla\varphi = \int_\Omega g\varphi. \end{cases}$$

Problem under study : coercivity ?

$$\forall \varphi \in H^1(\Omega)\,, \quad \underbrace{\int_\Omega \nabla \bar{u} \cdot \nabla \varphi - \int_\Omega \bar{u} \mathbf{V} \cdot \nabla \varphi}_{a(\bar{u},\varphi)} \;=\; \underbrace{\int_\Omega g \varphi}_{L(\varphi)}\,.$$

Coercivity ?

$$\begin{aligned} a(u,u) &= \int_\Omega \nabla u \cdot \nabla u - \int_\Omega u \mathbf{V} \cdot \nabla u \\ &= \int_\Omega |\nabla u|^2 + \int_\Omega \operatorname{div} \mathbf{V} \frac{u^2}{2} - \int_{\partial\Omega} u^2 \mathbf{V} \cdot \mathbf{n}. \end{aligned}$$

No hypotheses on $\operatorname{div}(\mathbf{V})$ and $\mathbf{V} \cdot \mathbf{n}$

$$\Downarrow$$

the bilinear form is not coercive

References

- Droniou, 2002
  - Dirichlet, Fourier and mixed boundary conditions
  - existence of a unique solution
  - direct explicit estimates on the solution
- Droniou, Gallouet, 2002
  - convergence of finite volume schemes
- Droniou, Vázquez, to appear
  - Neumann boundary conditions
  - existence under assumption: $\displaystyle\int_\Omega g = 0$
  - no uniqueness: operator has a kernel
  - no direct estimates, proof *via* Fredholm alternative

Addition of a lower order term

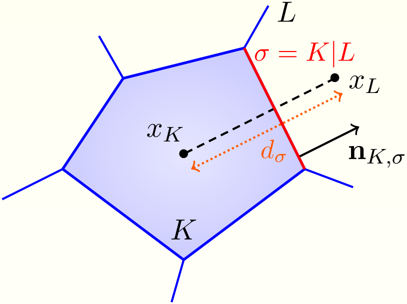
$$(\mathcal{P}_\gamma)\ \begin{cases} \mathrm{div}(\mathbf{J}) + \gamma u = g, \text{ with } \mathbf{J} = -\nabla u + \mathbf{V}u, \text{ in } \Omega \\ \mathbf{J}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \end{cases}$$

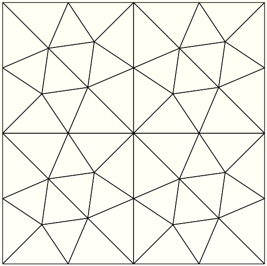
- $\gamma > 0$
- Direct a priori estimates on the solution (at least for large γ)
- Existence and uniqueness of a solution
- Use of Fredholm alternative to get any γ and $\gamma = 0$

Adaptation of this method to get convergence of some finite volume schemes for $(\mathcal{P}_0)$ and $(\mathcal{P}_\gamma)$?

Finite volume schemes

Admissible mesh : definition and notations

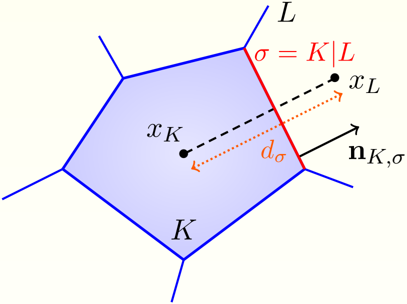




- $\mathcal{T}$: family of cells K, open convex polygonal subsets of Ω
- $\mathcal{E}$: family of edges σ
- $\mathcal{P}$: family of points $x_K \in K$ with $(x_K, x_L) \perp \sigma$
- For discrete Sobolev inequalities :

$$\exists \zeta > 0 \text{ such that } \mathrm{d}(x_K, \sigma) \geq \zeta \mathrm{d}_\sigma, \quad \forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K.$$

Principle of the scheme



Integration on each cell K of

$$\mathrm{div}\mathbf{J} = g,$$
$$\mathbf{J} = -\nabla u + \mathbf{V}u,$$

with

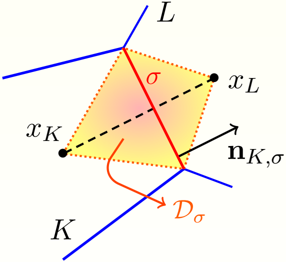
$$\mathbf{J}\cdot\mathbf{n} = 0 \text{ on all } \sigma \subset \Gamma.$$

- $\displaystyle\sum_{\sigma\in\mathcal{E}_{K,int}} \mathcal{F}_{K,\sigma} = \mathrm{m}(K) g_K$ for all $K \in \mathcal{T}$
- $\mathcal{F}_{K,\sigma}$ approximation of $\displaystyle\int_\sigma \mathbf{J}\cdot\mathbf{n}_{K,\sigma}$

Numerical fluxes

$$\mathcal{F}_{K,\sigma} \approx \int_\sigma \mathbf{J}\cdot\mathbf{n}_{K,\sigma} \text{ with } \mathbf{J} = -\nabla u + \mathbf{V} u$$

$$\Longrightarrow \quad \mathcal{F}_{K,\sigma} \approx \int_\sigma -\nabla u\cdot\mathbf{n}_{K,\sigma} + \int_\sigma u\mathbf{V}\cdot\mathbf{n}_{K,\sigma}$$



Approximation of $\mathbf{V}\cdot\mathbf{n}_{K,\sigma}$ on σ:

- $v_{K,\sigma} = \dfrac{1}{\mathrm{m}(\mathcal{D}_\sigma)}\displaystyle\int_{\mathcal{D}_\sigma}\mathbf{V}\cdot\mathbf{n}_{K,\sigma}$,
- $v_{K,\sigma} = \dfrac{1}{\mathrm{m}(\sigma)}\displaystyle\int_{\sigma}\mathbf{V}\cdot\mathbf{n}_{K,\sigma}$,
- $v_{K,\sigma} = \mathbf{V}(x_\sigma)\cdot\mathbf{n}_{K,\sigma}$

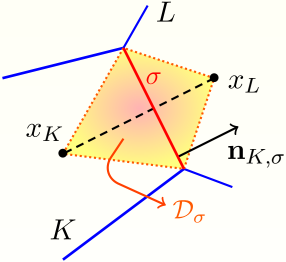
Centered fluxes

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}(u_K - u_L) + \mathrm{m}(\sigma) v_{K,\sigma}\frac{u_K + u_L}{2}.$$

Numerical fluxes

$$\mathcal{F}_{K,\sigma} \approx \int_\sigma \mathbf{J}\cdot\mathbf{n}_{K,\sigma} \text{ with } \mathbf{J} = -\nabla u + \mathbf{V}u$$

$$\Longrightarrow \quad \mathcal{F}_{K,\sigma} \approx \int_\sigma -\nabla u\cdot\mathbf{n}_{K,\sigma} + \int_\sigma u\mathbf{V}\cdot\mathbf{n}_{K,\sigma}$$



Approximation of $\mathbf{V}\cdot\mathbf{n}_{K,\sigma}$ on σ:

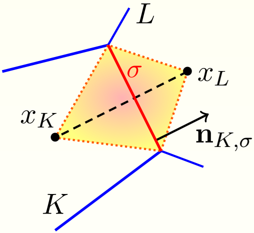
- $v_{K,\sigma} = \dfrac{1}{\mathrm{m}(\mathcal{D}_\sigma)}\displaystyle\int_{\mathcal{D}_\sigma}\mathbf{V}\cdot\mathbf{n}_{K,\sigma},$
- $v_{K,\sigma} = \dfrac{1}{\mathrm{m}(\sigma)}\displaystyle\int_{\sigma}\mathbf{V}\cdot\mathbf{n}_{K,\sigma},$
- $v_{K,\sigma} = \mathbf{V}(x_\sigma)\cdot\mathbf{n}_{K,\sigma}$

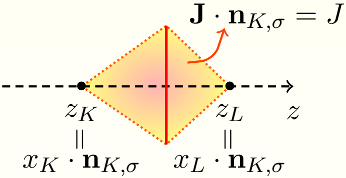
Upwind fluxes

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}(u_K - u_L) + \mathrm{m}(\sigma)(v^+_{K,\sigma}u_K - v^-_{K,\sigma}u_L).$$

Numerical fluxes

$$\mathcal{F}_{K,\sigma} \approx \int_\sigma \mathbf{J} \cdot \mathbf{n}_{K,\sigma} \text{ with } \mathbf{J} = -\nabla u + \mathbf{V} u$$





- Scharfetter, Gummel, 1969

$\Longrightarrow$ Resolution of the following ODE in 1D:

$$\begin{cases} -\dfrac{\partial u}{\partial z}(z) + v_{K,\sigma} u(z) = J, \ z \in [z_K, z_L], \\ u(z_K) = u_K. \end{cases}$$

$$\Longrightarrow \quad u(z) = \frac{J}{v_{K,\sigma}} + (u_K - \frac{J}{v_{K,\sigma}}) e^{v_{K,\sigma}(z - z_K)}.$$

Numerical fluxes

$$\mathcal{F}_{K,\sigma} \approx \int_\sigma \mathbf{J} \cdot \mathbf{n}_{K,\sigma} \text{ with } \mathbf{J} = -\nabla u + \mathbf{V} u$$

$$u(z) = \frac{J}{v_{K,\sigma}} + (u_K - \frac{J}{v_{K,\sigma}})e^{v_{K,\sigma}(z-z_K)}$$

Scharfetter-Gummel fluxes

$\mathcal{F}_{K,\sigma} = \mathrm{m}(\sigma)J$ is defined by imposing $u(z_L) = u_L$:

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B_{sg}(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B_{sg}(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)$$

where B_{sg} is the Bernoulli function:

$$B_{sg}(s) = \frac{s}{e^s - 1}.$$

Generic form of the fluxes

Scharfetter-Gummel fluxes

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B_{sg}(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B_{sg}(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)$$

with $B_{sg}(s) = \dfrac{s}{e^s - 1}$.

Centered fluxes

$$\begin{aligned}\mathcal{F}_{K,\sigma} = & \ \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}(u_K - u_L) + \mathrm{m}(\sigma)v_{K,\sigma}\frac{u_K + u_L}{2} \\ = & \ \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B_{ce}(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B_{ce}(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)\end{aligned}$$

with

$$B_{ce}(s) = 1 - \frac{s}{2}.$$

Generic form of the fluxes

Scharfetter-Gummel fluxes

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B_{sg}(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B_{sg}(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)$$

with $B_{sg}(s) = \dfrac{s}{e^s - 1}$.

Upwind fluxes

$$\begin{aligned}\mathcal{F}_{K,\sigma} = \quad & \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}(u_K - u_L) + \mathrm{m}(\sigma)(v^+_{K,\sigma}u_K - v^-_{K,\sigma}u_L) \\ = \quad & \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B_{up}(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B_{up}(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)\end{aligned}$$

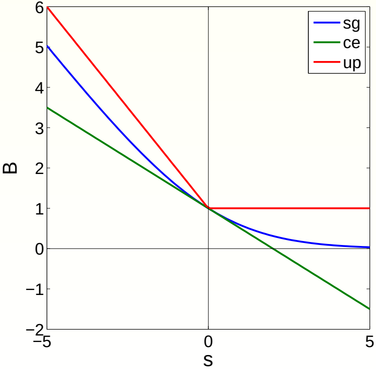
with

$$B_{up}(s) = 1 + s^-.$$

Generic form of the fluxes

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)$$

$$B_{sg}(s) = \frac{s}{e^s - 1}, \quad B_{ce}(s) = 1 - \frac{s}{2}, \quad B_{up}(s) = 1 + s^-.$$



Numerical schemes

Scheme for $(\mathcal{P}_0)$: $(\mathcal{S}_0)$

- $$\sum_{\sigma\in\mathcal{E}_{K,int}} \mathcal{F}_{K,\sigma} = \mathrm{m}(K) g_K$$
- $$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)$$

Properties of the function B

- B is Lipschitz-continuous on $\mathbb{R}$,
- $B(0) = 1$ and $B(s) > 0$ for all $s \in \mathbb{R}$ ($s < 2$ for B_{ce}),
- $B(s) - B(-s) = -s$ for all $s \in \mathbb{R}$

Numerical schemes

Scheme for $(\mathcal{P}_\gamma)$: $(\mathcal{S}_\gamma)$

- $$\sum_{\sigma\in\mathcal{E}_{K,int}} \mathcal{F}_{K,\sigma} + \gamma \mathrm{m}(K) u_K = \mathrm{m}(K) g_K$$
- $$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)$$

Properties of the function B

- B is Lipschitz-continuous on $\mathbb{R}$,
- $B(0) = 1$ and $B(s) > 0$ for all $s \in \mathbb{R}$ ($s < 2$ for B_{ce}),
- $B(s) - B(-s) = -s$ for all $s \in \mathbb{R}$

Particularity of the SG scheme

Case where $\mathbf{V} = \nabla\Phi$

$$\begin{cases} \mathrm{div}(\mathbf{J}) = 0, \text{ with } \mathbf{J} = -\nabla u + \nabla\Phi\ u, \text{ in } \Omega \\ \\ \mathbf{J}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \end{cases}$$

Kernel spanned by $\widehat{\overline{u}} = e^{\Phi}$

New definition of $v_{K,\sigma}$

$$v_{K,\sigma} = \frac{\Phi(x_L) - \Phi(x_K)}{\mathrm{d}_\sigma}$$

$$\Downarrow$$

$$\mathcal{F}_{K,\sigma} = \quad \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B_{sg}(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B_{sg}(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)$$

Particularity of the SG scheme

Case where $\mathbf{V} = \nabla\Phi$

$$\begin{cases} \mathrm{div}(\mathbf{J}) = 0, \ \text{with } \mathbf{J} = -\nabla u + \nabla\Phi \ u, \ \text{in } \Omega \\ \mathbf{J}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \end{cases}$$

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$$\Downarrow$$

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B_{sg}(\Phi(x_K) - \Phi(x_L))u_K - B_{sg}(\Phi(x_L) - \Phi(x_K))u_L\Big)$$

Particularity of the SG scheme

Case where $\mathbf{V} = \nabla\Phi$

$$\begin{cases} \mathrm{div}(\mathbf{J}) = 0, \ \text{with } \mathbf{J} = -\nabla u + \nabla\Phi\ u, \ \text{in } \Omega \\ \\ \mathbf{J}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \end{cases}$$

Kernel spanned by $\widehat{\overline{u}} = e^{\Phi}$

New definition of $v_{K,\sigma}$

$$v_{K,\sigma} = \frac{\Phi(x_L) - \Phi(x_K)}{\mathrm{d}_\sigma}$$

$$\Downarrow$$

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(\frac{\Phi(x_K) - \Phi(x_L)}{e^{\Phi(x_K)-\Phi(x_L)} - 1}u_K - \frac{\Phi(x_L) - \Phi(x_K)}{e^{\Phi(x_L)-\Phi(x_K)} - 1}u_L\Big)$$

Particularity of the SG scheme

Case where $\mathbf{V} = \nabla\Phi$

$$\begin{cases} \mathrm{div}(\mathbf{J}) = 0, \text{ with } \mathbf{J} = -\nabla u + \nabla\Phi\ u, \text{ in } \Omega \\[2ex] \mathbf{J}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \end{cases}$$

Kernel spanned by $\widehat{\overline{u}} = e^{\Phi}$

New definition of $v_{K,\sigma}$

$$v_{K,\sigma} = \frac{\Phi(x_L) - \Phi(x_K)}{\mathrm{d}_\sigma}$$

$$\Downarrow$$

$$\begin{aligned} \mathcal{F}_{K,\sigma} = {} & \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(\frac{\Phi(x_K) - \Phi(x_L)}{e^{\Phi(x_K)-\Phi(x_L)} - 1}u_K - \frac{\Phi(x_L) - \Phi(x_K)}{e^{\Phi(x_L)-\Phi(x_K)} - 1}u_L\Big) \\ = {} & 0 \quad \text{if } u_K = e^{\Phi(x_K)} \quad \forall K \in \mathcal{T} \end{aligned}$$

Existence and uniqueness of a solution

Scheme $(\mathcal{S}_0)$ under matricial form

- $\displaystyle\sum_{\sigma\in\mathcal{E}_{K,int}} \mathcal{F}_{K,\sigma} = \mathrm{m}(K) g_K$
- $\displaystyle\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)$

Matricial form

$$\mathbb{A}U = G$$

with

$$\mathbb{A}_{K,K} = \sum_{\sigma\in\mathcal{E}_{K,\mathrm{int}}} \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} B(-v_{K,\sigma}\mathrm{d}_\sigma)\,, \quad K \in \mathcal{T},$$

$$\mathbb{A}_{K,L} = -\frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} B(v_{K,\sigma}\mathrm{d}_\sigma)\,, \quad K \in \mathcal{T}\,, L \in N(K), \sigma = K|L,$$

$$\mathbb{A}_{K,L} = 0\,, \quad K \in \mathcal{T}\,, L \notin N(K).$$

Scheme $(\mathcal{S}_\gamma)$ under matricial form

- $$\sum_{\sigma\in\mathcal{E}_{K,int}} \mathcal{F}_{K,\sigma} + \gamma \mathrm{m}(K) u_K = \mathrm{m}(K) g_K$$
- $$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B(-v_{K,\sigma}\mathrm{d}_\sigma)u_K - B(v_{K,\sigma}\mathrm{d}_\sigma)u_L\Big)$$

Matricial form

$$\boxed{\mathbb{A}_\gamma U = G}$$

with

$$\mathbb{A}_\gamma = \mathbb{A} + \gamma\mathbb{D}$$
$$\mathbb{D} = \mathrm{Diag}(\mathrm{m}(K))$$

Properties of $\mathbb{A}$ and $\mathbb{A}_\gamma$

$$\mathbb{A}_{K,K} = \sum_{\sigma \in \mathcal{E}_{K,\mathrm{int}}} \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} B(-v_{K,\sigma}\mathrm{d}_\sigma),$$

$$\mathbb{A}_{K,L} = -\frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} B(v_{K,\sigma}\mathrm{d}_\sigma)\,, \text{ or } \mathbb{A}_{K,L} = 0.$$

Properties

- Strict positivity of the diagonal terms of $\mathbb{A}$ and $\mathbb{A}_\gamma$ $(\gamma > 0)$
- Nonpositivity of the extra diagonal terms
- $\mathbb{A}_{K,K} = -\sum_{L \in N(K)} \mathbb{A}_{L,K}\,, \quad \forall K \in \mathcal{T}.$

↪ $\mathbb{A}_\gamma$ $(\gamma > 0)$ is strictly diagonally dominant with respect to the columns.

↪ $\mathbb{A}_\gamma$ $(\gamma > 0)$ is an M-matrix and then invertible.

Kernel and Image of $\mathbb{A}$

Properties

- $\mathbb{A}_{K,K} = - \sum_{L \in N(K)} \mathbb{A}_{L,K} \, , \quad \forall K \in \mathcal{T}.$

↪ $(1, 1, \cdots, 1)^* \in \mathrm{Ker}(\mathbb{A}^*)$

↪ $\mathrm{Ker}(\mathbb{A}^*)$ and $\mathrm{Ker}(\mathbb{A})$ have at least dimension 1.

- $\mathrm{Im}(\mathbb{A}) = \left\{ (G_K)_{K \in \mathcal{T}} \; ; \; \sum_{K \in \mathcal{T}} G_K = 0 \right\}.$

Furthermore

- $\mathrm{Ker}(\mathbb{A})$ has dimension 1.
- If $U \in \mathrm{Ker}(\mathbb{A}) \backslash \{0\}$, then either $U > 0$ or $U < 0$.

Results

Theorem (C.C.-H. – J. Droniou)

- Existence and uniqueness of a solution to $(\mathcal{S}_\gamma)$
- The kernel of scheme $(\mathcal{S}_0)$ has dimension 1
  and is spanned by a function $\hat{u} = (\hat{u}_K)_{K\in\mathcal{T}} > 0$.
- If $\displaystyle\int_\Omega g = 0$, there exists a unique solution to $(\mathcal{S}_0)$ such that

$$\int_\Omega u = 0.$$

Results of convergence

Convergence of the scheme $(\mathcal{S}_\gamma)$

Theorem (C.C.-H. – J. Droniou)

- $(\mathcal{M}_n)$ sequence of admissible meshes (ζ not depending on n),
- $\mathrm{size}(\mathcal{M}_n) \to 0$ as $n \to \infty$,
- u_n unique solution to $(\mathcal{S}_\gamma)$.

Then,

- $u_n \to \bar{u}$ in $L^2(\Omega)$,
- $\bar{u} \in H^1(\Omega)$ is the unique weak solution to $(\mathcal{P}_\gamma)$.

Sketch of the proof

Estimates for large γ (u solution to $(\mathcal{S}_\gamma)$)

$\exists \gamma_0 > 0$ and $C > 0$ such that for all $\gamma \geq \gamma_0$

$$\underbrace{|u|^2_{1,\mathcal{M}}}_{\text{discrete } H^1-\text{norm}} + \|u\|^2_{L^2(\Omega)} \leq \frac{C}{\gamma}\|u\|^2_{L^2(\Omega)}$$

Passing to the limit in the scheme $(\mathcal{S}_\gamma)$

- u_n unique solution to $(\mathcal{S}_\gamma)$,
- $(|u_n|_{1,\mathcal{M}_n} + ||u_n||_{L^2(\Omega)})_{n\geq 1}$ bounded,
- $u_n \to \bar{u}$ in $L^2(\Omega)$ as $n \to \infty$,with $\bar{u} \in H^1(\Omega)$

Then, $\bar{u}$ is a weak solution to $(\mathcal{P}_\gamma)$

$\Longrightarrow$ **Estimates for any $\gamma \geq 0$** (by contradiction)

Convergence of the kernel for the scheme $(\mathcal{S}_0)$

Theorem (C.C.-H. – J. Droniou)

- $(\mathcal{M}_n)$ sequence of admissible meshes (ζ not depending on n)
- $\mathrm{size}(\mathcal{M}_n) \to 0$ as $n \to \infty$,
- $\widehat{u}_n$ unique positive element with L^2-norm equal to 1 in the kernel of $(\mathcal{S}_0)$

Then,

- $\widehat{u}_n \to \widehat{\overline{u}}$ in $L^2(\Omega)$
- $\widehat{\overline{u}} \in H^1(\Omega)$ is the unique positive element in the kernel of $(\mathcal{P}_0)$ with L^2-norm equal to 1.

Convergence of the scheme $(\mathcal{S}_0)$

Theorem (C.C.-H. – J. Droniou)

- $\displaystyle\int_\Omega g = 0$,
- $(\mathcal{M}_n)$ sequence of admissible meshes (ζ not depending on n)
- $\mathrm{size}(\mathcal{M}_n) \to 0$ as $n \to \infty$,
- u_n unique solution to $(\mathcal{S}_0)$ with zero mean value.

Then,

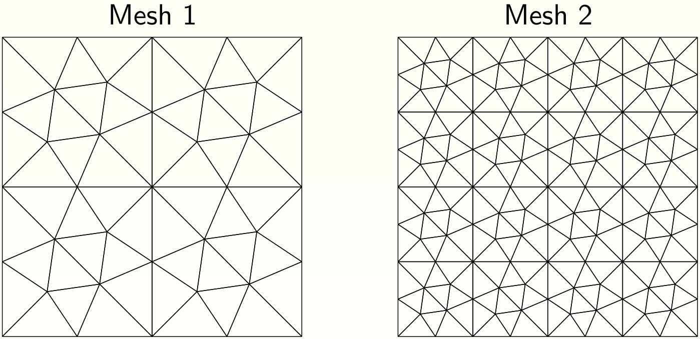
- $u_n \to \bar{u}$ in $L^2(\Omega)$
- $\bar{u} \in H^1(\Omega)$ is the unique solution to $(\mathcal{P}_0)$ with zero mean value.

Numerical experiments

General framework

Domain: $\Omega = [0,1] \times [0,1]$.

Meshes: Sequence of 7 admissible triangular meshes



N triangles → $4N$ triangles

size : h → size : $\dfrac{h}{2}$

Test case 1

$$\mathbf{V}(x,y) = \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \nabla\Phi \quad \text{with} \quad \Phi(x,y) = 10x.$$

$$\left.\begin{array}{l} \bullet\ \text{Kernel of } (\mathcal{P}_0) \Rightarrow \widehat{\widehat{u}}(x,y) = \exp(10x), \\ \bullet\ \text{Kernel of } (\mathcal{S}_0) \Rightarrow \widehat{u}, \end{array}\right\} \begin{array}{l} \text{normalized to 1} \\ \text{in } L^2\text{-norm} \end{array}$$

Numerical convergence (different choices for $v_{K,\sigma}$ coincide)

Number of triangles	$\Vert\widehat{u} - \widehat{\widehat{u}}\Vert_{L^2(\Omega)}$ centered scheme	$\Vert\widehat{u} - \widehat{\widehat{u}}\Vert_{L^2(\Omega)}$ upwind scheme	$\Vert\widehat{u} - \widehat{\widehat{u}}\Vert_{L^2(\Omega)}$ SG scheme
56	4.48e-02	1.66e-01	5.73e-16
224	1.26e-02	1.05e-01	8.28e-16
896	3.14e-03	5.88e-02	8.48e-15
3584	7.51e-04	3.04e-02	6.84e-15
14336	1.84e-04	1.55e-02	2.35e-14
57344	4.85e-05	7.83e-03	6.26e-14
229376	1.14e-05	3.94e-03	6.78e-14

Test case 2

$$\mathbf{V}(x,y) = \nabla\Phi \text{ with } \Phi(x,y) = \log(x+y-2xy).$$

- Kernel of $(\mathcal{P}_0)$ spanned by $\widehat{\overline{u}}(x,y) = x+y-2xy$.

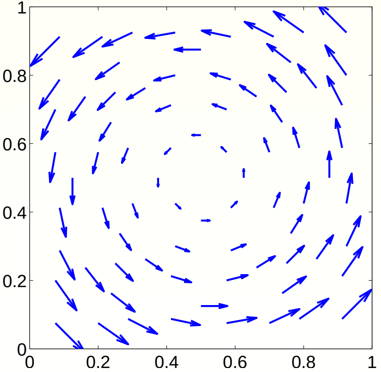
Numerical convergence

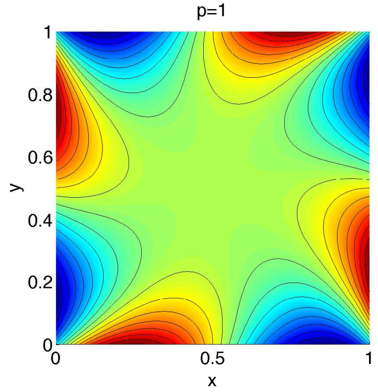
- $v_{K,\sigma} = \mathbf{V}(x_\sigma)\cdot\mathbf{n}_{K,\sigma}$
  - Order 2 in L^2-norm for the centered and the SG schemes,
  - Order 1 in L^2-norm for the upwind scheme
- $v_{K,\sigma} = \dfrac{\Phi(x_L)-\Phi(x_K)}{\mathrm{d}_\sigma}$
  - SG scheme is "exact"

Test case 3

$$\mathbf{V}(x,y) = 10^p \begin{pmatrix} -(y-0.5) \\ (x-0.5) \end{pmatrix}$$

Numerical solutions (SG scheme, Mesh 7)

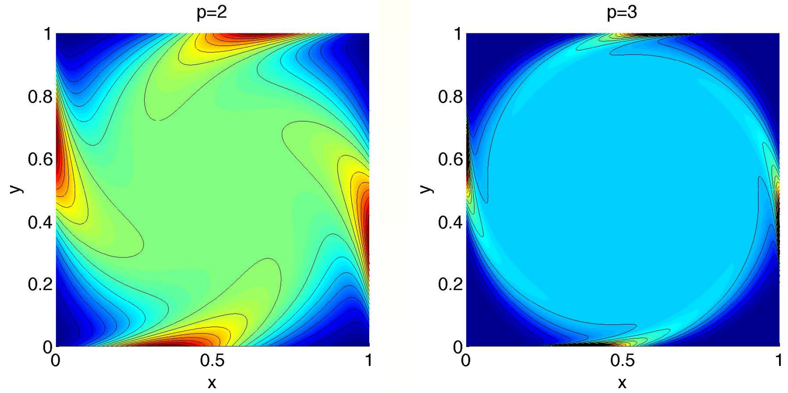




Test case 3

$$\mathbf{V}(x,y) = 10^p \begin{pmatrix} -(y-0.5) \\ (x-0.5) \end{pmatrix}$$

Numerical solutions (SG scheme, Mesh 7)



Test case 3

Positivity of the kernel

Mesh	Centered scheme	Upwind scheme	SG scheme
	min	min	min
1	-1.56e-02	2.15e-01	2.03e-01
2	-7.86e-02	4.41e-02	3.47e-02
3	-2.20e-01	2.62e-03	1.15e-03
4	-7.70e-02	4.67e-05	5.09e-06
5	-2.77e-03	7.94e-07	6.50e-09
6	-1.07e-09	1.82e-08	1.24e-10
7	1.00e-11	9.44e-10	2.34e-11

Péclet condition for the centered scheme

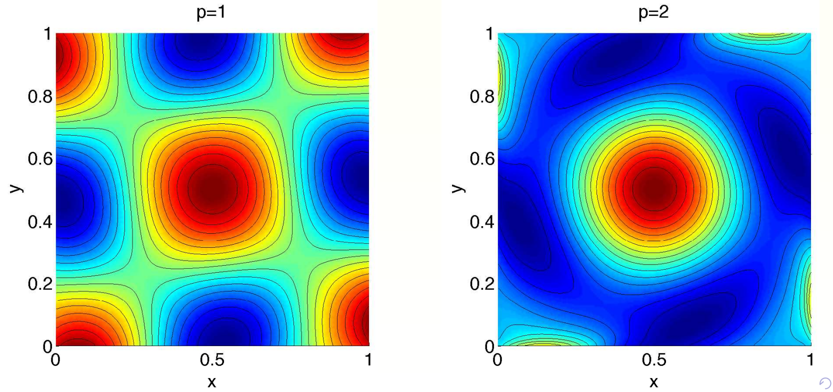
$$|v_{K,\sigma}\mathrm{d}_\sigma| < 2 \quad \text{for all } \sigma.$$

Test case 4 (with a right hand side g)

$$\mathbf{V}(x,y) = 10^p \begin{pmatrix} -(y-0.5) \\ \\ (x-0.5) \end{pmatrix}$$

$$g(x,y) = \cos(2\pi x)\cos(2\pi y)$$

Numerical solution

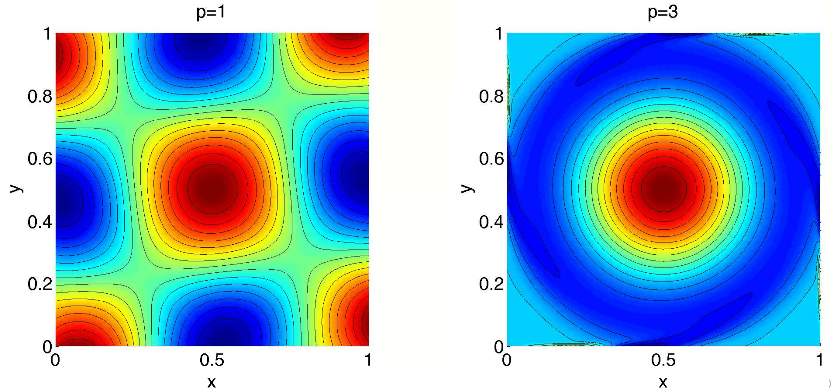


Test case 4 (with a right hand side g)

$$\mathbf{V}(x,y) = 10^p \begin{pmatrix} -(y-0.5) \\ \\ (x-0.5) \end{pmatrix}$$

$$g(x,y) = \cos(2\pi x)\cos(2\pi y)$$

Numerical solution



Test case 5 (g + nonhomogeneous boundary conditions h)

$$\mathbf{V}(x,y) = -100 \begin{pmatrix} x+y \\ y-x \end{pmatrix} \quad (\mathrm{div}\,\mathbf{V} < 0)$$

Right hand side and boundary conditions

chosen such that $\bar{u}(x,y) = 30x(1-x)y(1-y)$ ($\|\bar{u}\|_{L^2(\Omega)} = 1$)

Numerical convergence

- Order 2 in L^2-norm for the centered and the SG schemes,
- Order 1 in L^2-norm for the upwind scheme