Stabilized Finite Element Method for 3D Navier-Stokes Equations with Physical Boundary Conditions

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Mathematical Framework

2D : stationary incompressible Navier-Stokes eqs.

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Boundary conditions: $$\begin{cases} \mathbf{u}\cdot\mathbf{n} = 0 & , \quad \mathbf{u}\cdot\mathbf{t} = \mathbf{0} \quad on\ \Gamma_1 \\ \widetilde{p} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} = p_0 & , \quad \mathbf{u}\cdot\mathbf{t} = \mathbf{0} \quad on\ \Gamma_2 \\ \mathbf{u}\cdot\mathbf{n} = 0 & , \quad \omega = \omega_0 \qquad on\ \Gamma_3 \end{cases}$$

with $\omega = curl\mathbf{u}$ the scalar vorticity

(see also *Conca et al*. IJNMF '95, *Dubois* M3AS '02)

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Amara, Capatina and Trujillo Math. Comp. ’07 :
Three-fields formulation in $(\mathbf{u}, \omega, p)$ thanks to :

$$\mathbf{u}.\nabla\mathbf{u} = \omega\mathbf{u}^{\perp} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u})$$

Mathematical Framework

From now on : $\Omega \subset \mathbb{R}^3$ connected bounded polyhedron.
Stationary incompressible Navier-Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla\widetilde{p} = \mathbf{f} & in\ \Omega, \\ div\mathbf{u} = 0 & in\ \Omega. \end{cases}$$

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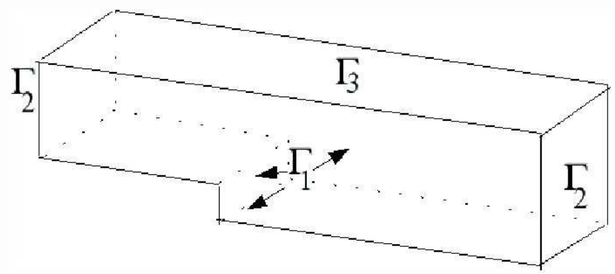

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We take: $\mathbf{f} \in \mathbf{L}^{\frac{4}{3}}(\Omega),\ \omega_0 = 0$, $p_0 = 0$ and $|\Gamma_2| > 0$.

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By means of the relation : $\mathbf{u}\cdot\nabla u + \nabla\widetilde{p} = \nabla p + \omega \times \mathbf{u}$,
the system becomes:

$$\left\{\begin{array}{ll} \nu\mathbf{curl}\omega + \nabla p + \omega \times \mathbf{u} = \mathbf{f} & in\ \Omega, \\ \omega = \mathbf{curl}\,\mathbf{u} & in\ \Omega, \\ div\mathbf{u} = 0 & in\ \Omega. \end{array}\right.$$

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Unknowns:

- vector fields: $\mathbf{u}$, ω
- scalar field: p.

The linear Stokes operator

Associated Stokes problem :

$$\begin{cases} \nu \mathbf{curl}\omega + \nabla p = \mathbf{g} & in\ \Omega, \\ \omega = \mathbf{curl u} & in\ \Omega, \\ div\mathbf{u} = 0 & in\ \Omega, \end{cases}$$

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Hypothesis: $\mathbf{g} \in \mathbf{L}^{\frac{4}{3}}(\Omega)$.

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Mixed variational formulation:

$$\left\{ \begin{array}{l} Find \quad (\sigma, \mathbf{u}) \in \mathbf{X} \times \mathbf{M} \; such\, that \\ \;a(\sigma, \tau) + b(\tau, \mathbf{u}) = 0 \quad \forall \tau \in \mathbf{X}, \\ \;b(\sigma, \mathbf{v}) = -l(\mathbf{v}) \qquad\qquad \forall \mathbf{v} \in \mathbf{M}, \end{array} \right.$$

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for all $\sigma = (\omega, p)$, $\tau = (\theta, q) \in \mathbf{X}$ and $\mathbf{v} \in \mathbf{M}$:

$$\begin{aligned} a(\sigma, \tau) &= \nu \int_\Omega \omega . \theta d\Omega, \\ b(\tau, \mathbf{v}) &= -\nu \int_\Omega \theta . \mathbf{curl} \mathbf{v} d\Omega + \int_\Omega q \, div \mathbf{v} d\Omega, \\ l(\mathbf{v}) &= \int_\Omega \mathbf{g} . \mathbf{v} d\Omega. \end{aligned}$$

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The following Hilbert spaces are employed :

$$\mathbf{M} = \{\mathbf{v} \in \mathbf{H}(div, \mathbf{curl}; \Omega); \quad \mathbf{v}\cdot\mathbf{n}|_{\Gamma_1 \cup \Gamma_3} = 0, \mathbf{v}\times\mathbf{n}|_{\Gamma_1 \cup \Gamma_2} = 0\},$$

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$\mathbf{H}(div, \mathbf{curl}; \Omega)$ and $\mathbf{M}$ are both normed by

$$\|\mathbf{v}\|_{\mathbf{M}} = (\|\mathbf{v}\|^2_{0,\Omega} + \|div\mathbf{v}\|^2_{0,\Omega} + \|\mathbf{curl}\mathbf{v}\|^2_{0,\Omega})^{1/2}.$$

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Last two assumptions hold if : $\mathbf{M} \subset \mathbf{H}^s(\Omega)$, $\frac{3}{4} < s \leq 1$

(in 2D, we can prove : $\mathbf{M} \subset \mathbf{H}^s(\Omega)$ with $\frac{1}{2} < s \leq 1$)

The linear Stokes operator

Continuous linear Stokes operator S defined by:

$$\begin{aligned} S:\ \mathbf{L}^{4/3}(\Omega) &\rightarrow \mathbf{X}\times\mathbf{L}^4(\Omega)\\ \mathbf{g} &\mapsto S(\mathbf{g}) = (\sigma,\mathbf{u}). \end{aligned}$$

The linear Stokes operator

Babuška-Brezzi:

$$\left\{ \begin{array}{ll} find \quad (\sigma, \mathbf{u}) \in \mathbf{X} \times \mathbf{M} \; such \, that & \\ a(\sigma, \tau) + b(\tau, \mathbf{u}) = 0 & \forall \tau \in \mathbf{X}, \\ b(\sigma, \mathbf{v}) = -l(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{M}, \end{array} \right.$$

admits a unique solution if:

- $$\inf_{\mathbf{v} \in \mathbf{M} \setminus \{0\}} \sup_{\sigma \in \mathbf{X}} \frac{b(\sigma, \mathbf{v})}{||\mathbf{v}||_{\mathbf{M}} ||\sigma||_{\mathbf{X}}} \geq \gamma > 0$$

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- $a(.,.)$ is $\mathbf{V}-$elliptic, where

$$\mathbf{V} = \{\tau \in \mathbf{X};\; b(\tau,\mathbf{v}) = 0, \forall \mathbf{v} \in \mathbf{M}\}$$

The nonlinear Navier-Stokes operator

We introduce the nonlinear operator

$$
\begin{aligned}
&G : \mathbf{X} \times \mathbf{L}^4(\Omega) \to \mathbf{L}^{4/3}(\Omega) \\
&\qquad G(\tau, \mathbf{v}) = \theta \times \mathbf{v}, \qquad\qquad where\ \tau = (\theta, q).
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Navier-Stokes equations:

$$F(\sigma, \mathbf{u}) = 0,$$

where F is defined by :

$$F : \mathbf{X} \times \mathbf{L}^4(\Omega) \to \mathbf{X} \times \mathbf{L}^4(\Omega)$$
$$F(\tau, \mathbf{v}) = (\tau, \mathbf{v}) - S(\mathbf{f} - G(\tau, \mathbf{v})).$$

The nonlinear Navier-Stokes operator

We assume that there exists a solution $(\sigma, \mathbf{u})$ such that:

$$F(\sigma, \mathbf{u}) = 0 \quad and \quad DF(\sigma, \mathbf{u}) \; is \; an \; isomorphism \; on \; \mathbf{X} \times \mathbf{L}^4(\Omega),$$

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where

$$DF(\sigma, \mathbf{u}) = Id + S(DG(\sigma, \mathbf{u})).$$

and

$$DG(\sigma, \mathbf{u})(\tau, \mathbf{v}) = \theta \times \mathbf{u} + \omega \times \mathbf{v} \qquad \forall \tau = (\theta, q) \in \mathbf{X}.$$

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- $\mathbf{M}_h = \{\mathbf{v}_h \in \mathbf{M}; \forall K \in \mathcal{T}_h,\ \mathbf{v}_h\,|_K \in P_1(K)\} \subset \mathcal{C}^0(\overline{\Omega})$,

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$$\mathbf{X}_h = \mathbf{L}_h \times L_h \quad and \quad \mathbf{M}_h$$

are discrete subspaces of $\mathbf{X}$ and $\mathbf{M}$.

The discrete Stokes operator S_h

$$\begin{aligned} S_h : \mathbf{L}^{\frac{4}{3}}(\Omega) &\longrightarrow \mathbf{X}_h \times \mathbf{M}_h \\ \mathbf{g} &\longmapsto (\sigma_h, \mathbf{u}_h) \end{aligned}$$

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where

- $A_h(\sigma_h, \tau_h) = \sum_{e \in \mathcal{E}_h} h_e \int_e [p_h][q_h] ds + \sum_{e \subset \Gamma_2} h_e \int_e p_h q_h d\Gamma,$
- $\beta > 0$ stabilization parameter,
- $[.]$ the jump across the edge $e \in \mathcal{E}_h$.

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$\Rightarrow$Existence and uniqueness for discrete Stokes pb.

The discrete Stokes operator

The operator S_h is linear, continuous and satisfies:

$$\|S_h(\mathbf{g})\|_{\mathbf{X}\times\mathbf{L}^4(\Omega)} \leq c\,\|\mathbf{g}\|_{\mathbf{L}^{4/3}(\Omega)}$$

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Moreover, if $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and $(\sigma, \mathbf{u}) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^2(\Omega)$:

$$\|(S - S_h)(\mathbf{g})\|_{\mathbf{X}\times\mathbf{M}} \leq ch\,\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}\,.$$

The discrete Stokes operator

Proof of the inf-sup condition: (usually difficult)
Since, for all $\tau_h = (\theta_h, q_h) \in \mathbf{X}_h$

$$b(\tau_h, \mathbf{v}_h) = -(\theta_h, \mathbf{curl}\, \mathbf{v}_h) + (q_h, div\, \mathbf{v}_h),$$

taking

$$\overline{\tau}_h = (-\mathbf{curl}\mathbf{v}_h, div\mathbf{v}_h) \in \mathbf{X}_h$$

we obtain

$$b(\overline{\tau}_h, \mathbf{v}_h) = \|\overline{\tau}_h\|^2_{\mathbf{X}} = \|\mathbf{curl}\mathbf{v}_h\|^2_{0,\Omega} + \|div\mathbf{v}_h\|^2_{0,\Omega} = |\mathbf{v}_h|^2_{\mathbf{M}}$$

and $\displaystyle\sup_{\tau_h \in \mathbf{X}_h} \frac{b(\tau_h, \mathbf{v}_h)}{\|\tau_h\|_{\mathbf{X}}} = \frac{b(\overline{\tau}_h, \mathbf{v}_h)}{\|\tau_h\|_{\mathbf{X}}} = \|\overline{\tau}_h\|_{\mathbf{x}} = |\mathbf{v}_h|_{\mathbf{M}}.$

Inf-Sup condition verified with constant $\gamma = 1.$

The discrete Stokes operator

Proof of the $\mathbf{V}_h$- coercivity (usually trivial) of a_h:

$$a_h = a + \beta A_h.$$

Semi-norm on $\mathbf{X}_h$ associated to A_h: for all $\tau_h \in \mathbf{X}_h$,

$$|\tau_h|_h = \sqrt{A_h(\tau_h, \tau_h)} = \Big(\sum_{e \in \mathcal{C}_h} h_e \, \|[q_h]\|^2_{0,\,e}\Big)^{\frac{1}{2}}$$

We have for all $\tau_h = (\theta_h, q_h) \in \mathbf{V}_h$:

$$a_h(\tau_h, \tau_h) = \|\theta_h\|^2_{0,\Omega} + \beta|\tau_h|^2_h \geq \alpha\|\tau_h\|^2_{\mathbf{X}}$$

Proof of the $\mathbf{X}_h$- continuity of a_h: for all $\tau_h \in \mathbf{X}_h$,

$$|\tau_h|_h \leq c\, \|q_h\|_{0,\Omega}\,.$$

The discrete Navier-Stokes problem

The discrete Navier-Stokes formulation can be written:

$$F_h(\sigma_h, \mathbf{u}_h) = (\mathbf{0}, \mathbf{0})$$

where F_h is defined by :

$$\begin{aligned} F_h : \mathbf{X} \times \mathbf{L}^4(\Omega) &\rightarrow \mathbf{X} \times \mathbf{L}^4(\Omega) \\ F_h(\tau, \mathbf{v}) = (\tau, \mathbf{v}) &- S_h(\mathbf{f} - G(\tau, \mathbf{v})). \end{aligned}$$

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The functional F_h is differentiable and:

$$DF_h(\sigma_h, \mathbf{u}_h) = Id + S_h(DG(\sigma_h, \mathbf{u}_h)).$$

Main tool: variant of the **implicit function theorem.**

The discrete Navier-Stokes problem

We show that: $\forall(\tau,\mathbf{v}) \in \mathbf{Y} = \mathbf{X} \times \mathbf{L}^4(\Omega)$,

- $\|DF_h(\overline{\tau},\overline{\mathbf{v}}) - DF_h(\tau,\mathbf{v})\| \leq c\,\|(\overline{\tau},\overline{\mathbf{v}}) - (\tau,\mathbf{v})\|_{\mathbf{Y}}$.
- $\lim_{h\to 0} \|F_h(\sigma,\mathbf{u})\|_{\mathbf{Y}} = 0.$
- $\exists h_0, \forall h \leq h_0, DF_h(\sigma,\mathbf{u})$ *is an isomorphism and* $\left\|DF_h(\sigma,\mathbf{u})^{-1}\right\| \leq 2\left\|DF(\sigma,\mathbf{u})^{-1}\right\|$.

The discrete Navier-Stokes problem

Then

- Uniqueness for $h < h_0$ in a neighborhood of $(\sigma, \mathbf{u})$.

The discrete Navier-Stokes problem

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$$\|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\|_{\mathbf{Y}} \leq c \, \|F_h(\sigma, \mathbf{u})\|_{\mathbf{Y}} \, .$$

The discrete Navier-Stokes problem

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- *a posteriori* estimates

$$\|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\|_{\mathbf{Y}} \leq c \, \|F(\sigma_h, \mathbf{u}_h)\|_{\mathbf{Y}} \, .$$

A priori estimates

We get :

- Unconditionally convergent method, since

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- Aubin-Nitsche argument $\Rightarrow \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^4(\Omega)} \leq O(h^{5/4})$ (respt. $O(h^{3/2})$ in 2D)

A posteriori estimators

Residuals on every element K of the triangulation:

A posteriori estimators

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$$\eta_K^2 = \|\eta_1\|_{0,K}^2 + \|\eta_2\|_{0,K}^2 + h_K^{2s} \|\eta_3\|_{0,K}^2 + h_e^{2s-1} \sum_{e \in \partial K} (\|\eta_4\|_{0,e}^2 + \|\eta_5\|_{0,e}^2)$$

where $s \in \left]\frac{3}{4}, 1\right]$ is such that $\mathbf{M} \subset \mathbf{H}^s(\Omega)$. Then :

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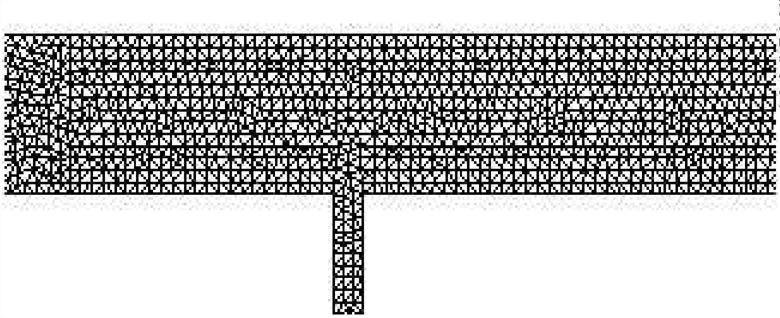
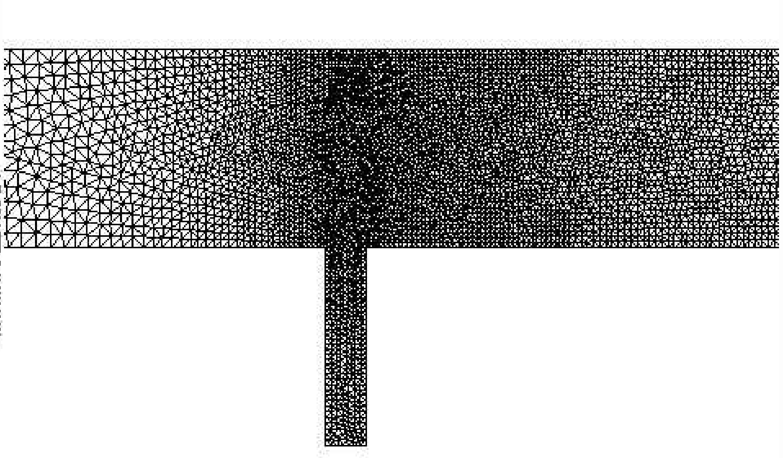
where $s \in \left]\frac{3}{4}, 1\right]$ is such that $\mathbf{M} \subset \mathbf{H}^s(\Omega)$. Then :

$$\|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\|_{\mathbf{Y}} \leq c(\sum_{K \in \mathcal{T}_h} \eta_K^2)^{1/2}.$$

A posteriori estimators: A 2D example

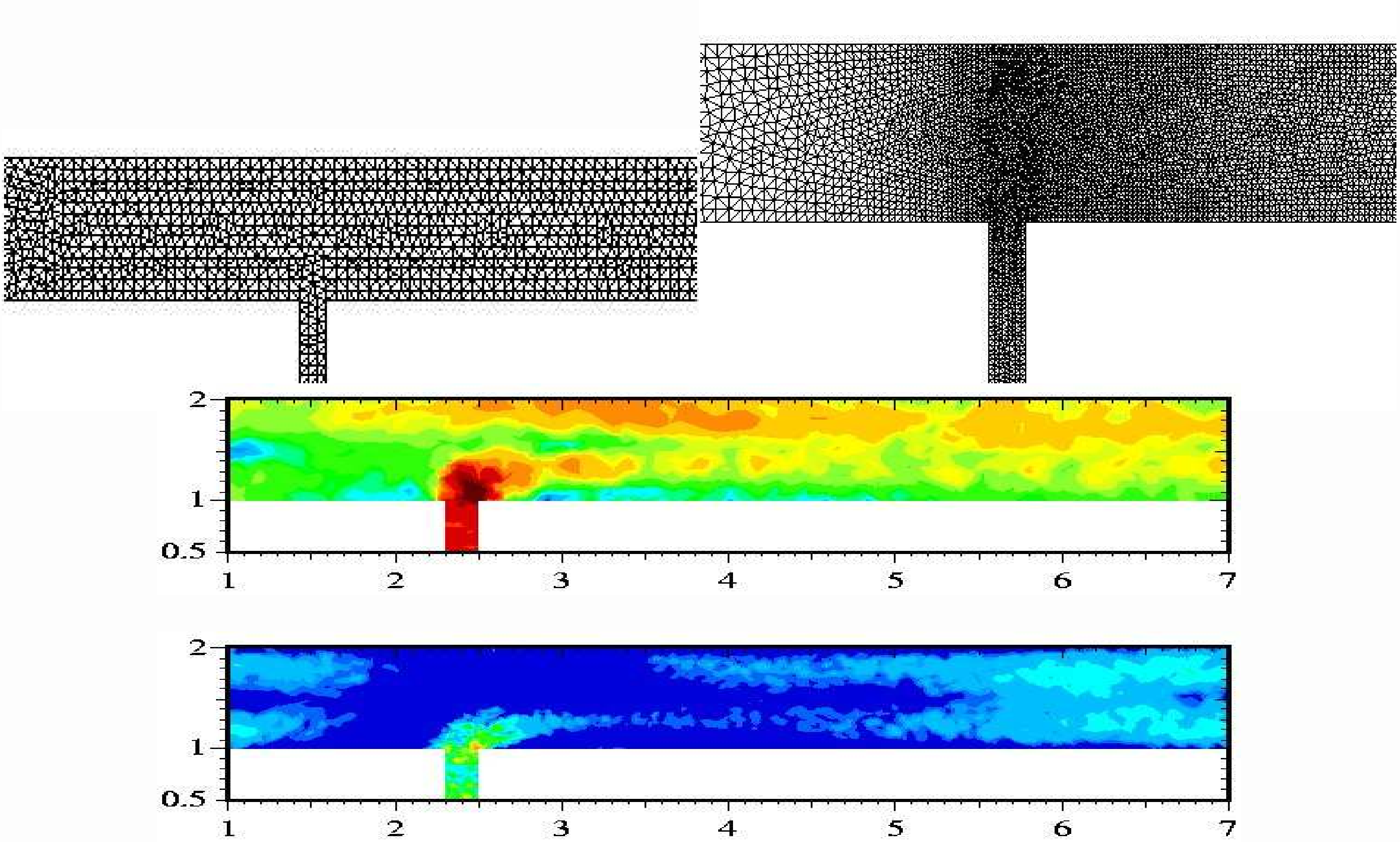


Coarse grid



Refined grid

A posteriori estimators: A 2D example



Estimators on the different meshes

A posteriori estimators

A tool to fix the parameter β

Example on the 2D step test

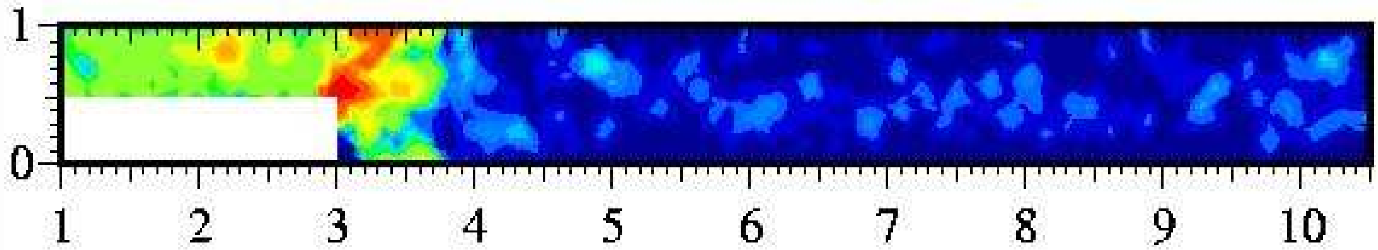
Values of η

A posteriori estimators

A tool to fix the parameter β

Example on the 2D step test

Values of η for $\beta = 0.03$,

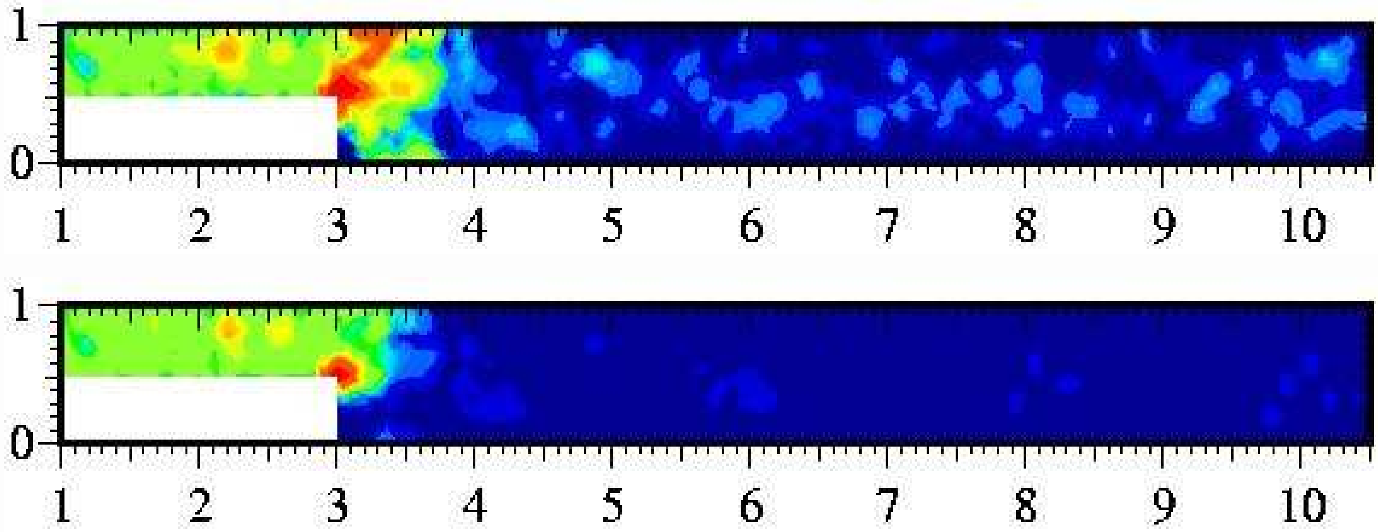


A posteriori estimators

A tool to fix the parameter β

Example on the 2D step test

Values of η for $\beta = 0.03$, $\beta = 0.5$

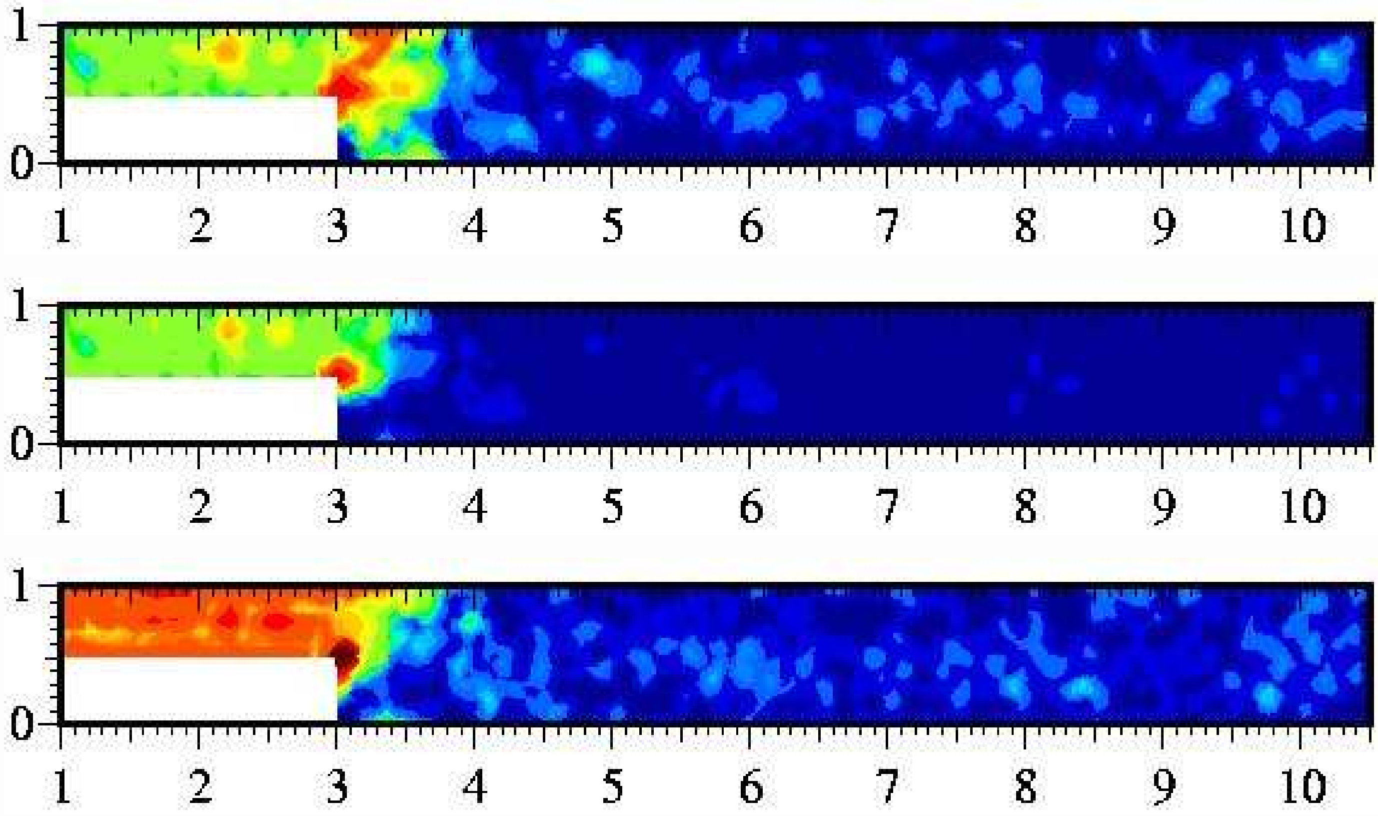


A posteriori estimators

A tool to fix the parameter β

Example on the 2D step test

Values of η for $\beta = 0.03$, $\beta = 0.5$ and $\beta = 2$



Numerical results: Convergence Rate

$$\Omega =]-1,1[^3,$$

$$p = sin(\pi x)sin(\pi y)sin(\pi z),\ u_1 = cos(\pi x)sin(\pi y)sin(\pi z),$$

$$u_2 = sin(\pi x)cos(\pi y)sin(\pi z),\ u_3 = -2sin(\pi x)sin(\pi y)cos(\pi z).$$

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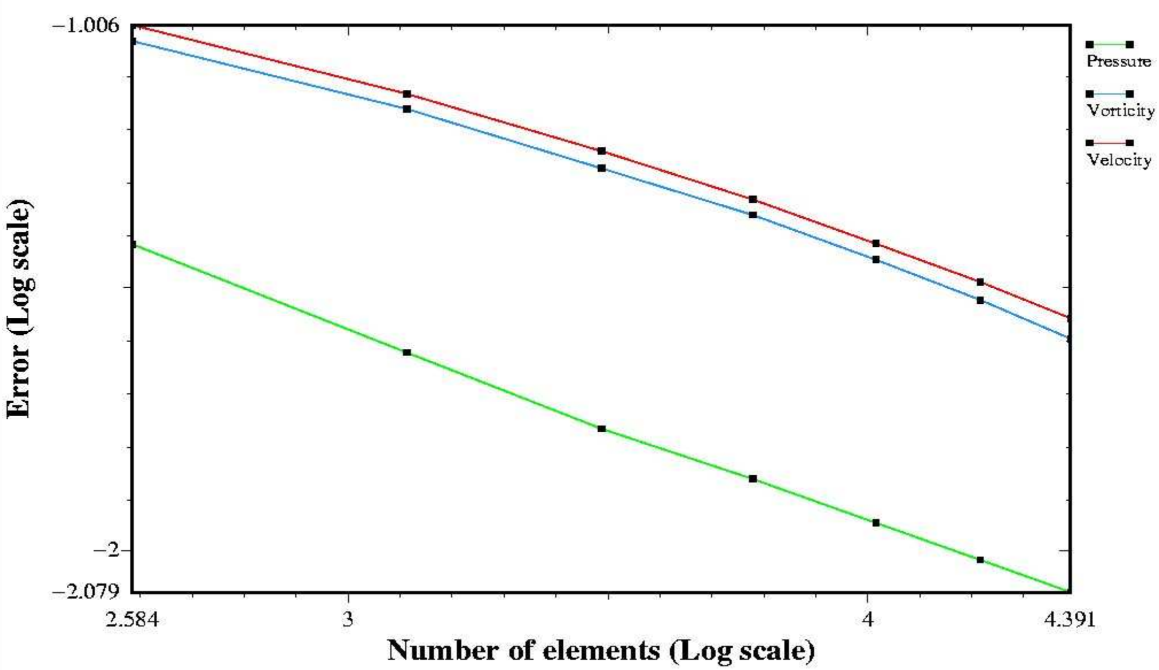

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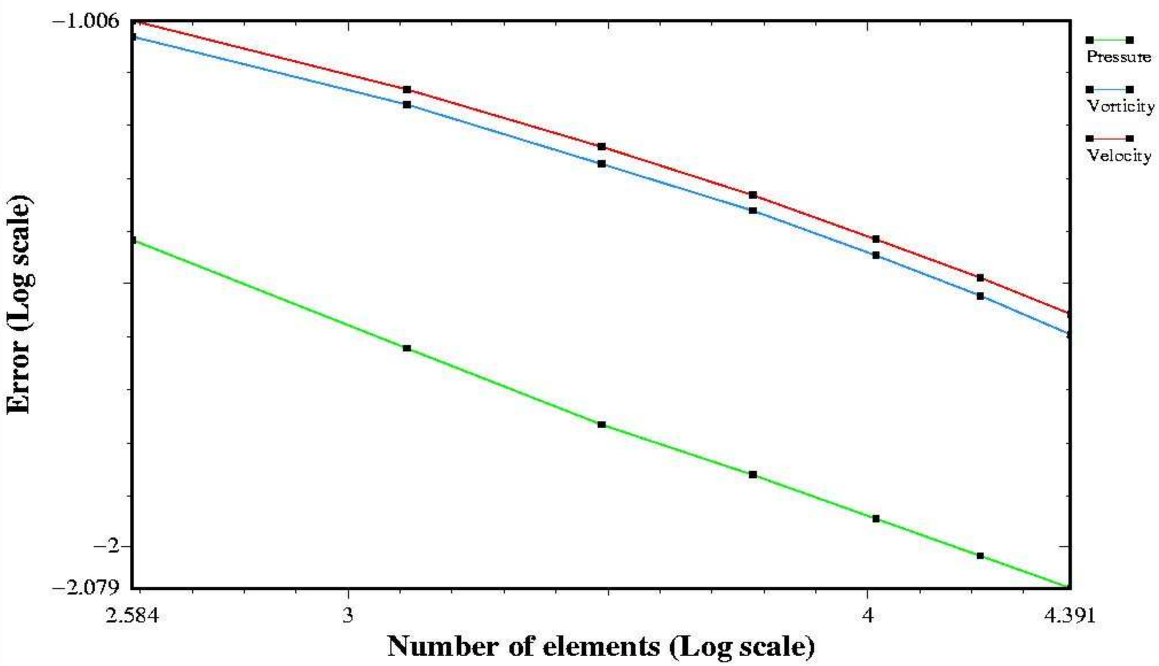
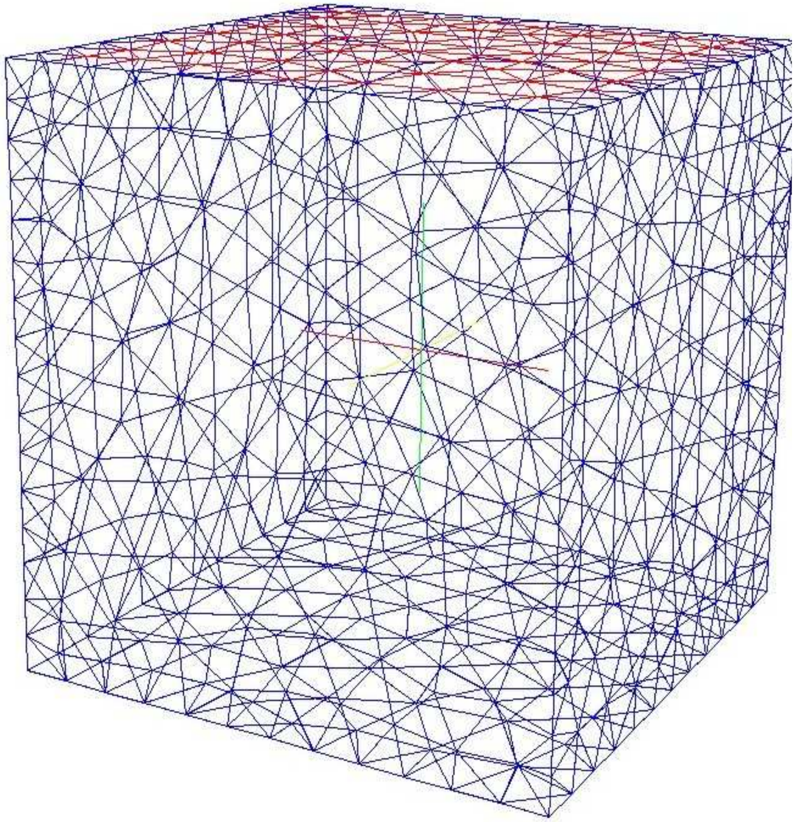


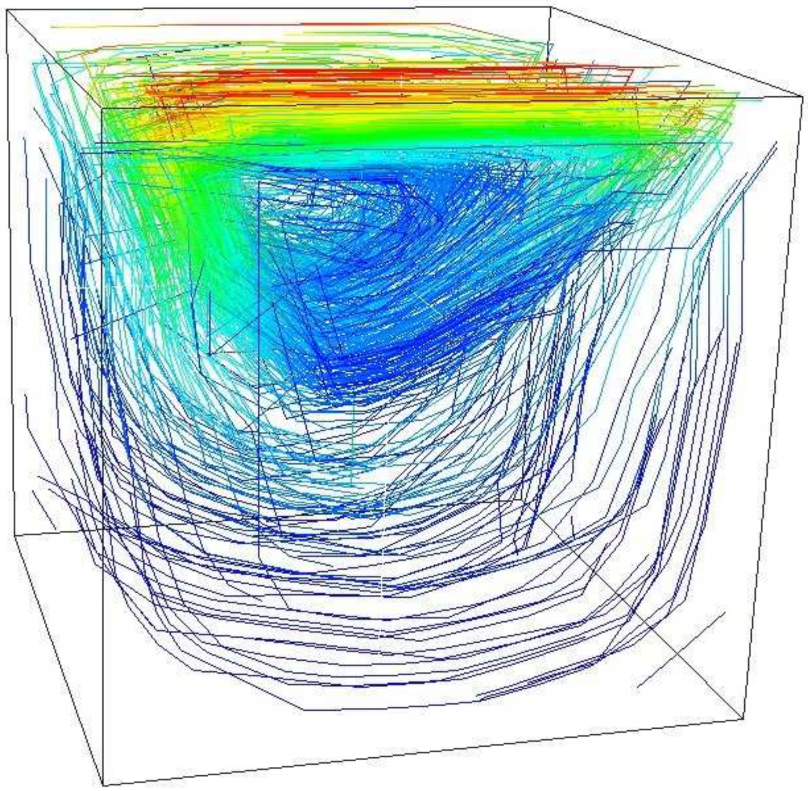
Slope :$-\frac{1}{3}$ and $h \sim N^{-\frac{1}{3}}$ then $errors \sim O(h)$

Numerical results: The cavity tests (Re=100)

The cavity test on the unit cube

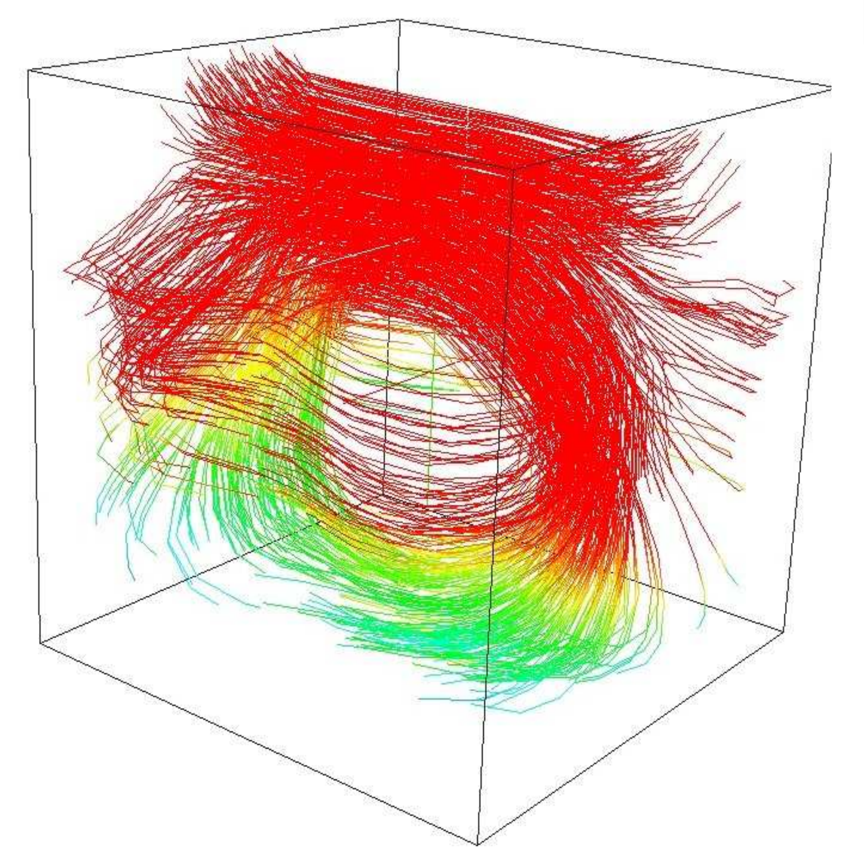


Mesh: 3232 elem., 838 nodes

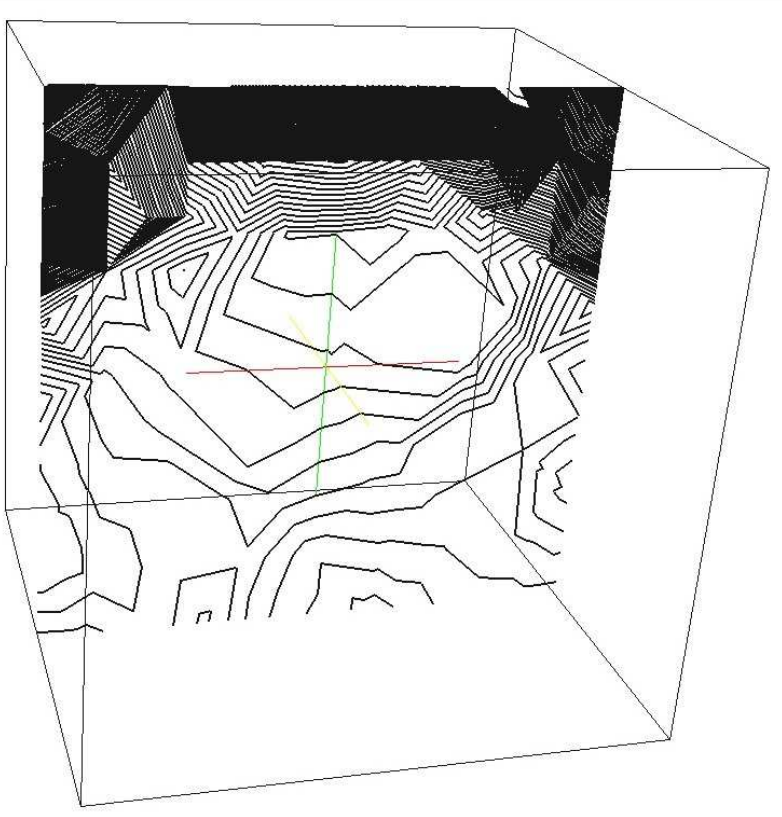


Streamlines

Numerical results: The cavity tests (Re=100)



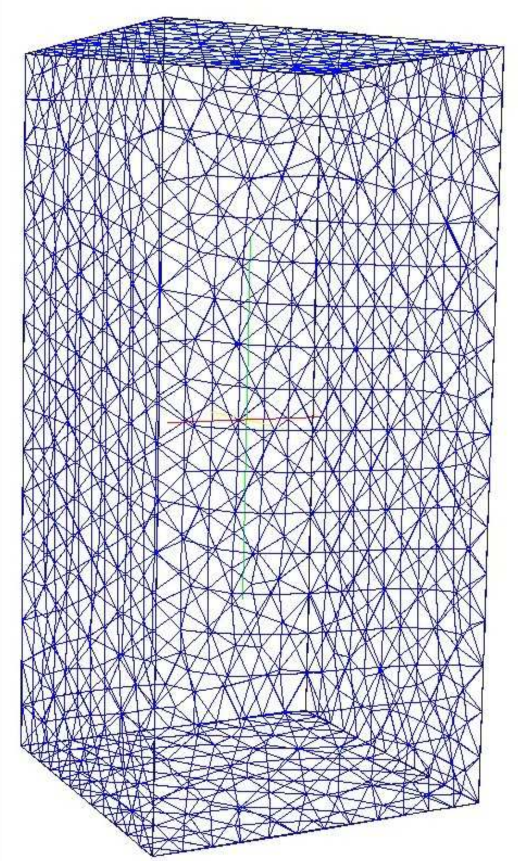
Vorticity lines



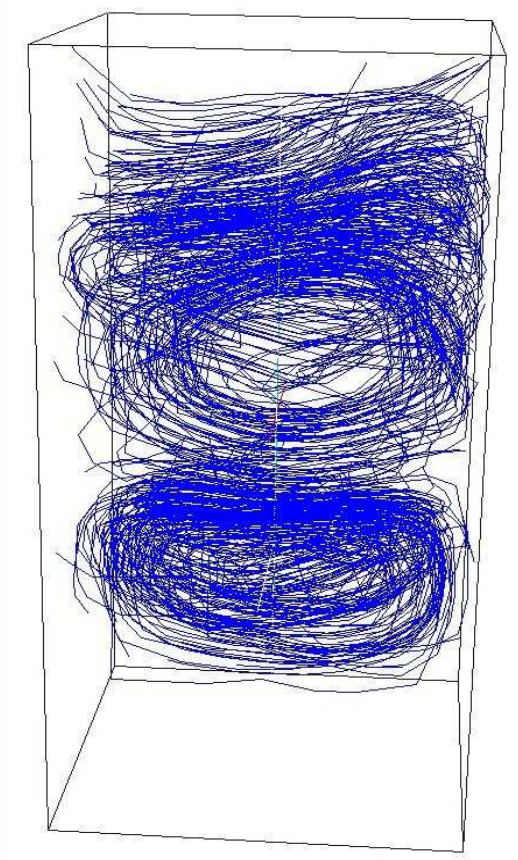
Pressure isolines

Numerical results: The cavity tests (Re=5000)

The cavity test on the domain $]0,1[\times]0,1[\times]0,2[$

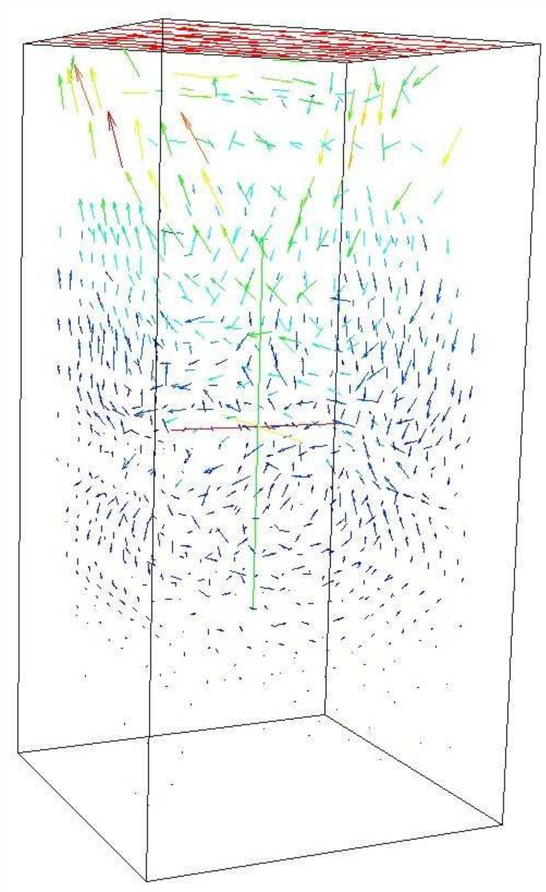


Mesh: 8619 elem., 1920 nodes

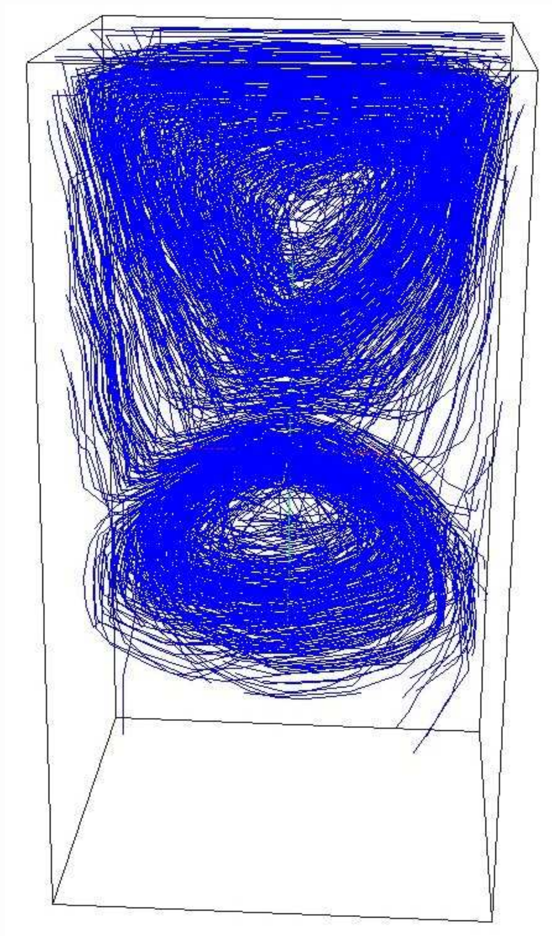


Vorticity lines

Numerical results: The cavity tests (Re=5000)

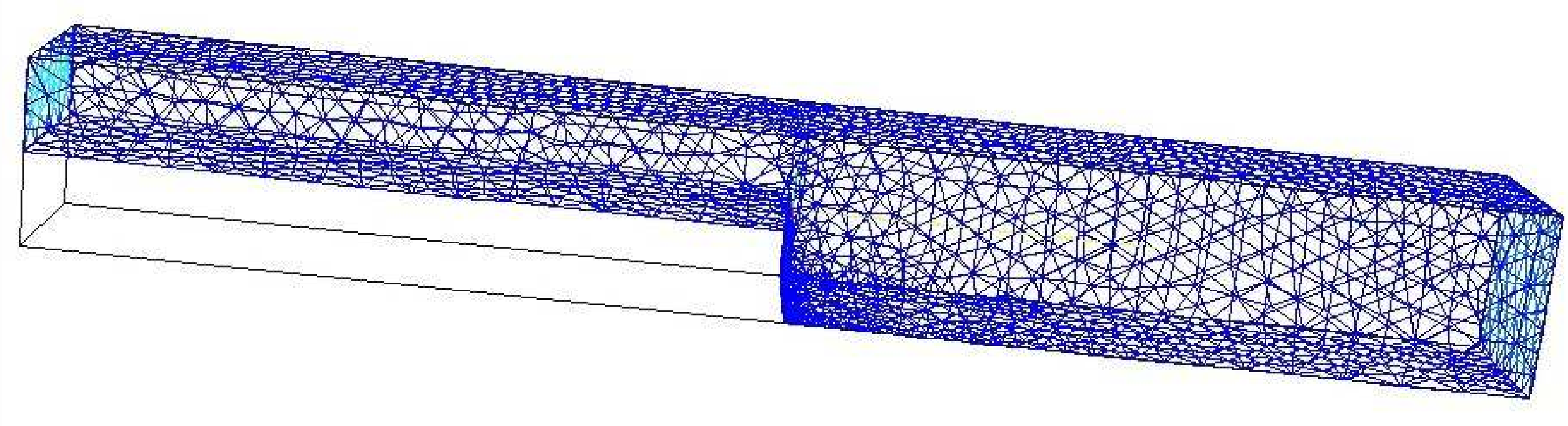


Velocity



Streamlines

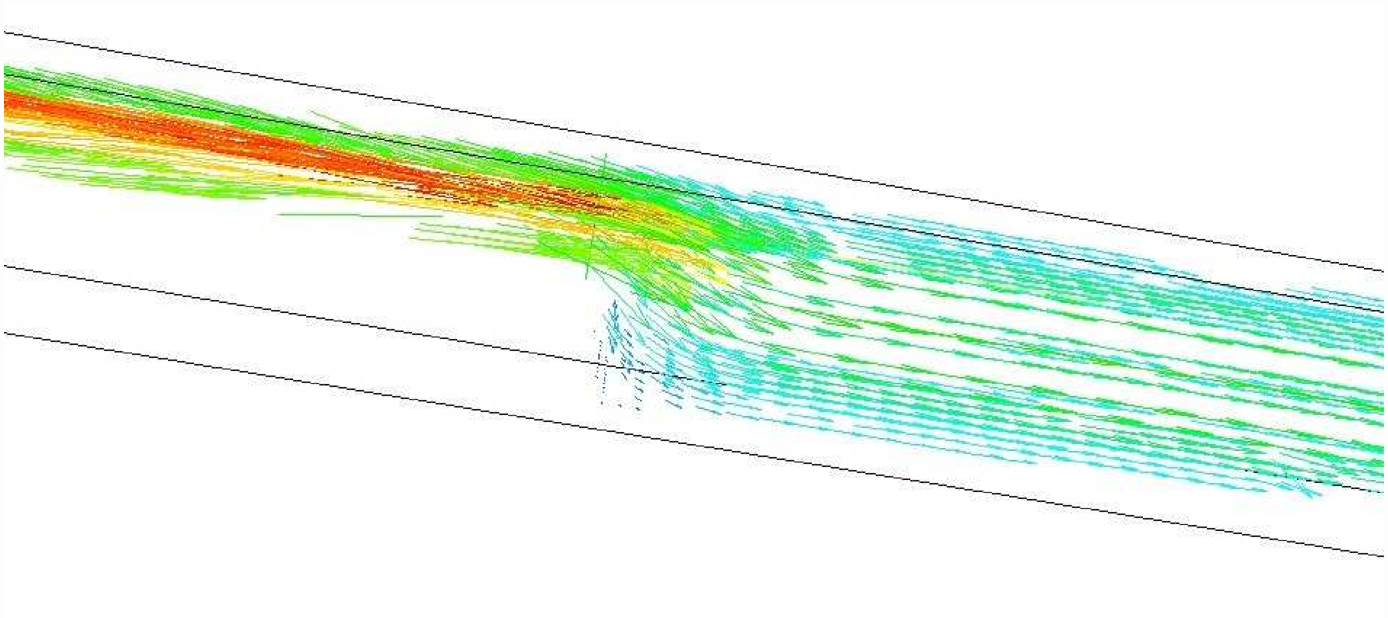
Numerical results: The step test (Re=10)



Mesh :10794 elements, 2665 nodes

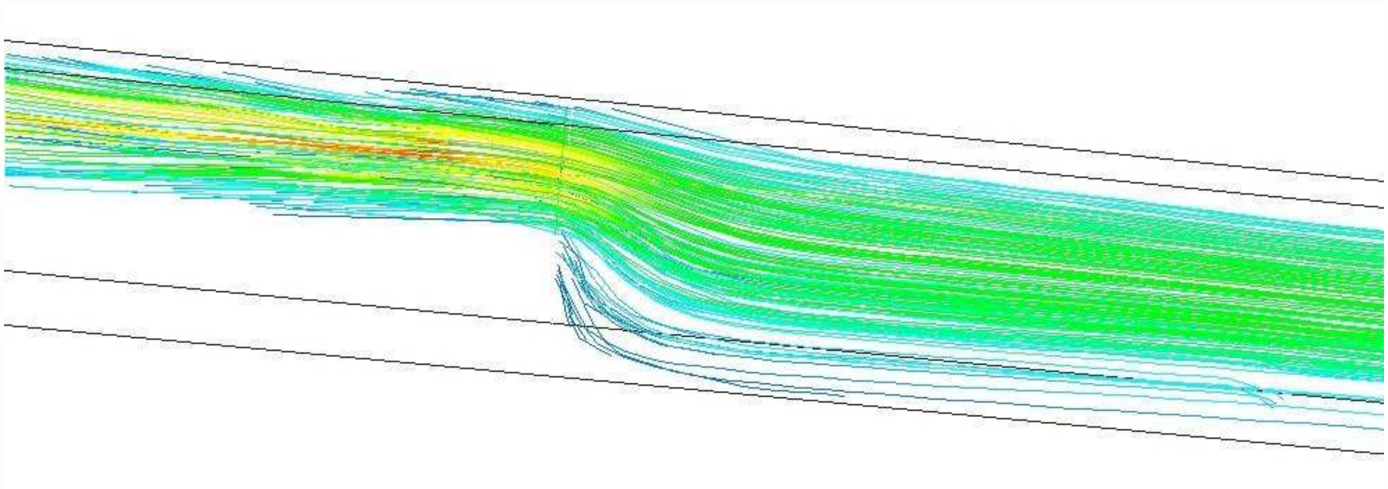
Pressure imposed on the inlet and outlet boundaries.

Numerical results: The step test (Re=10)



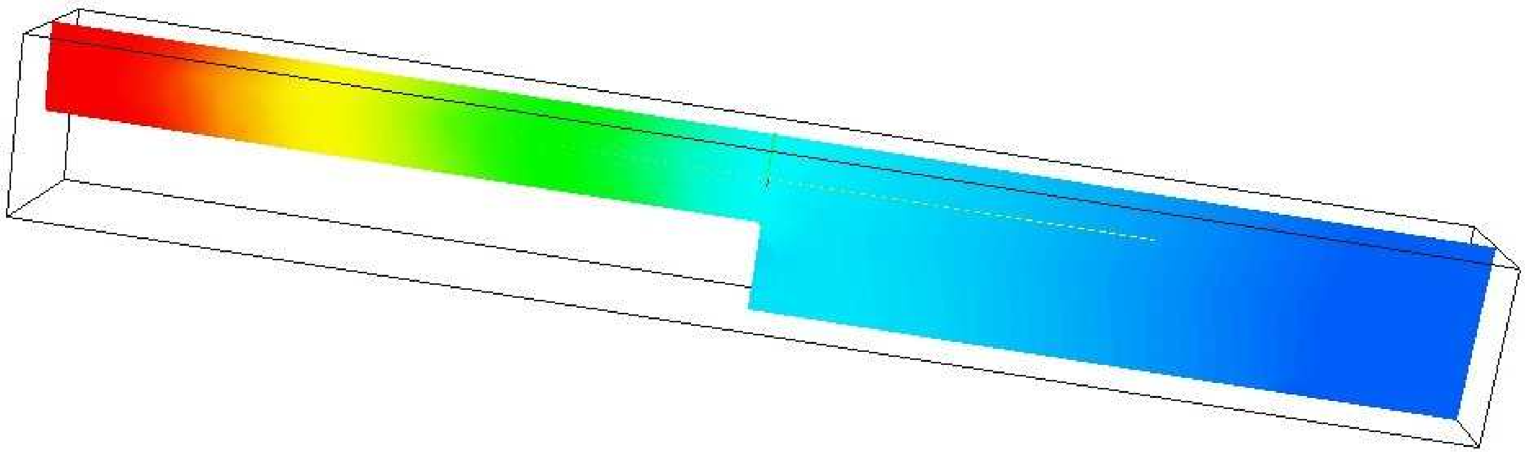
Velocity

Numerical results: The step test (Re=10)



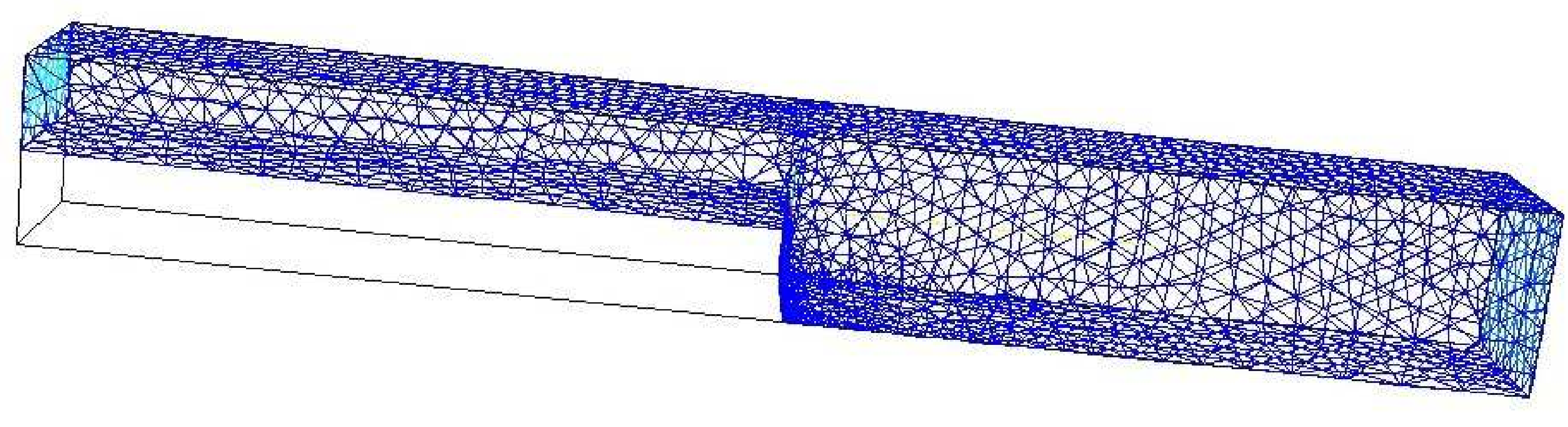
Streamlines

Numerical results: The step test (Re=10)



Pressure

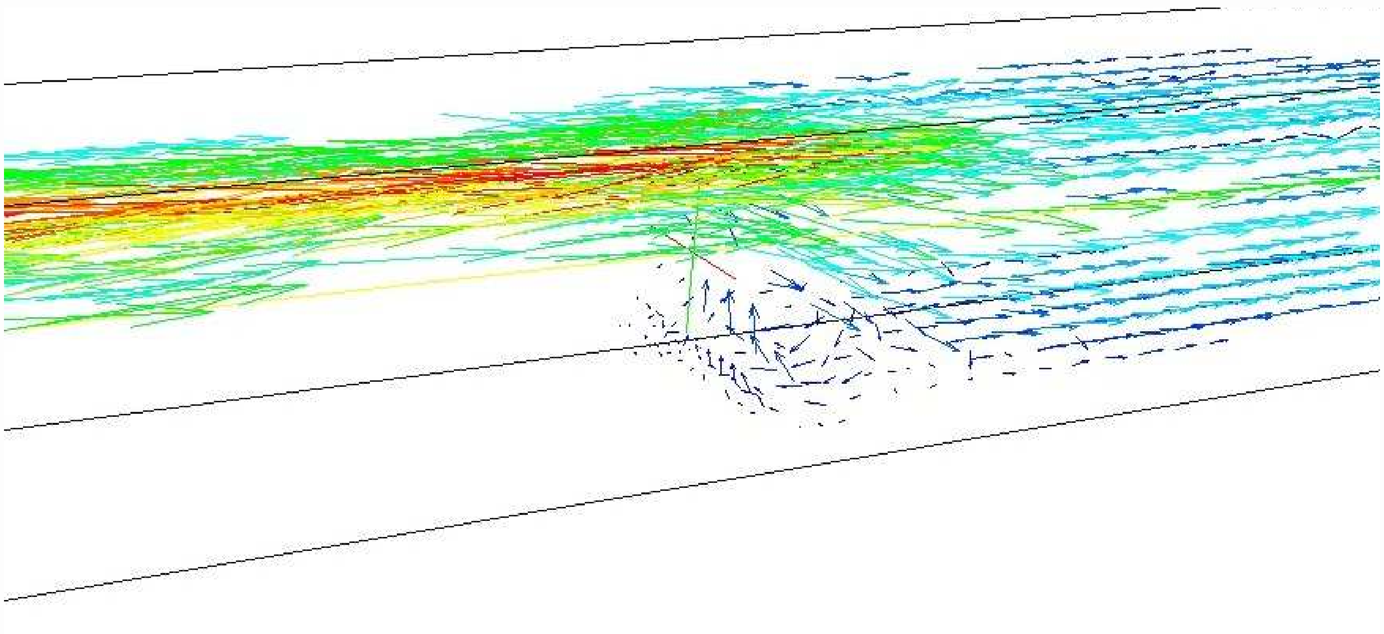
Numerical results: The step test (Re=1000)



Mesh :10794 elements, 2665 nodes

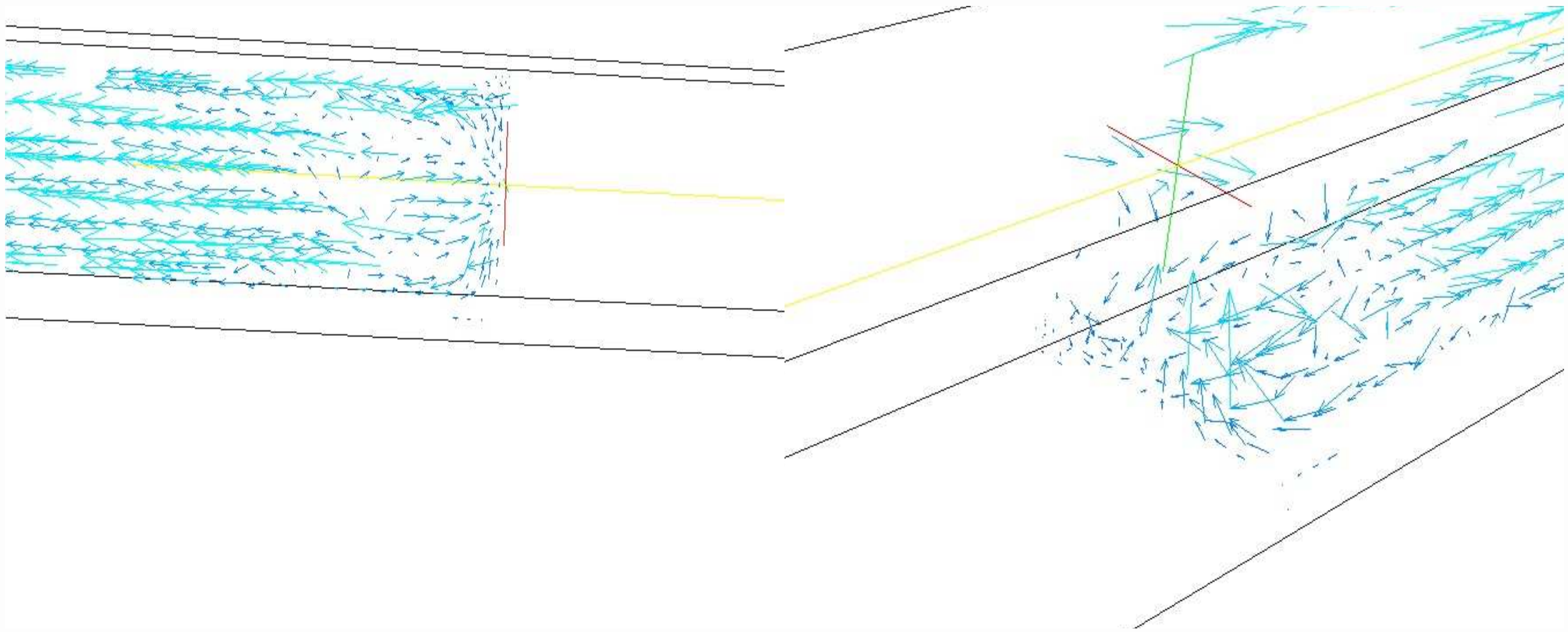
Pressure imposed on the inlet and outlet boundaries.

Numerical results: The step test (Re=1000)



Streamlines closed to the step

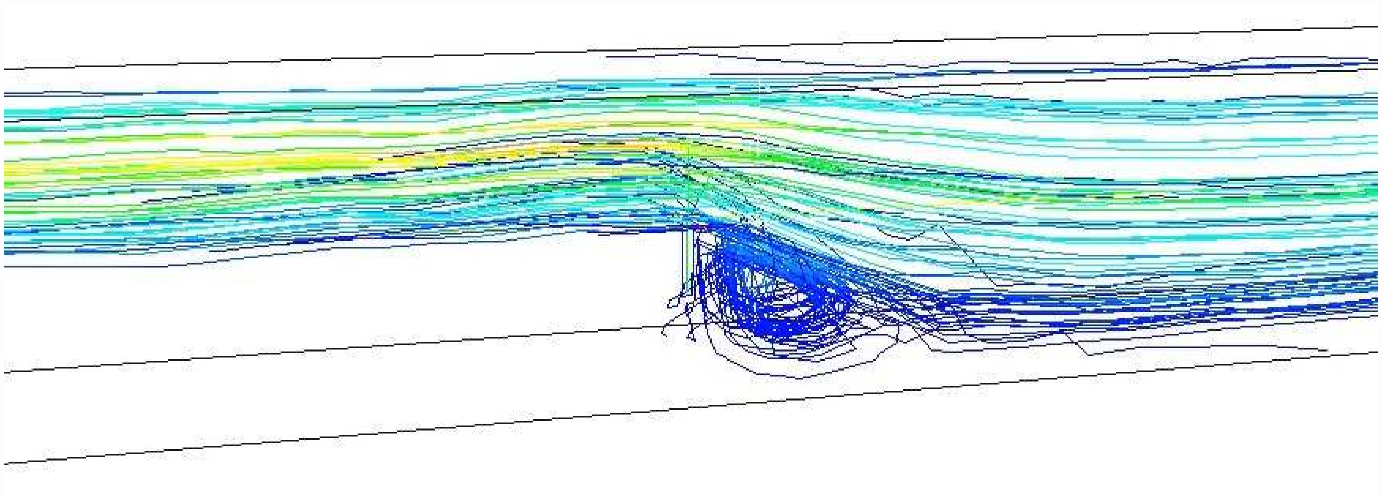
Numerical results: The step test (Re=1000)



View from the bottom | Lateral view

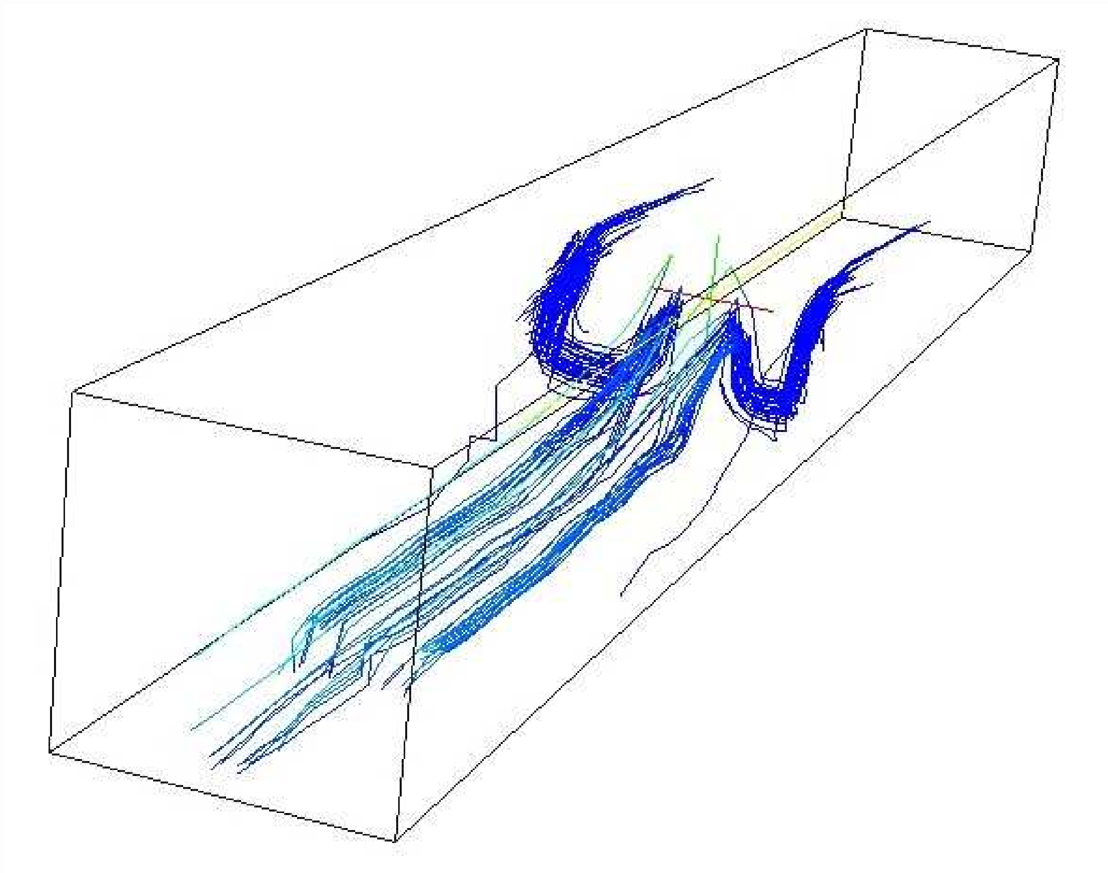
Velocity near the step

Numerical results: The step test (Re=1000)



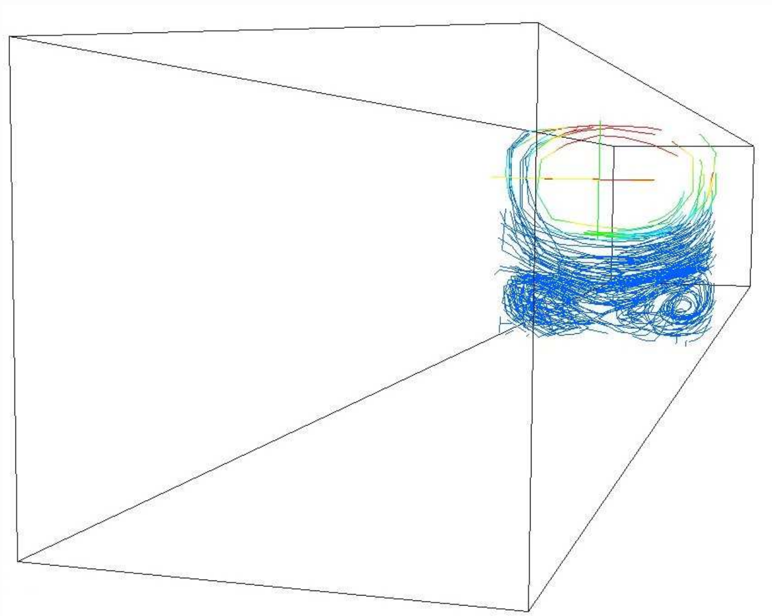
Streamlines closed to the step

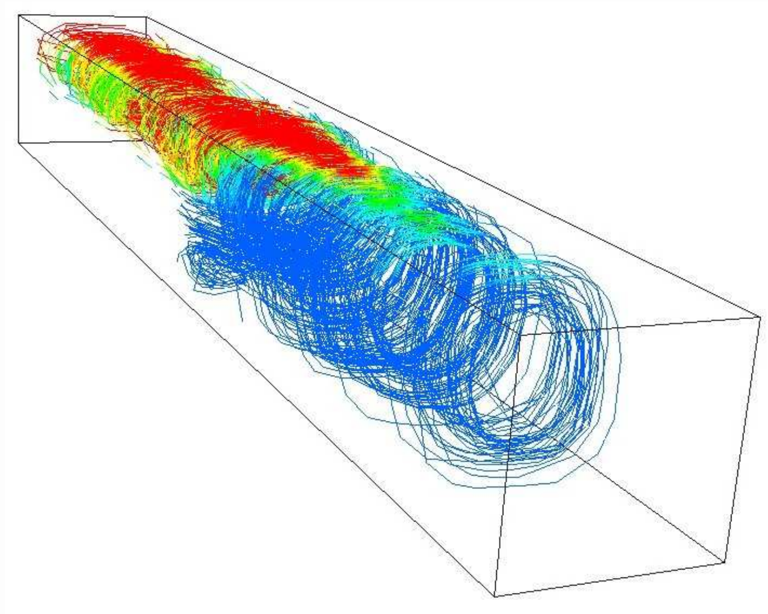
Numerical results: The step test (Re=1000)



Streamlines closed to the step

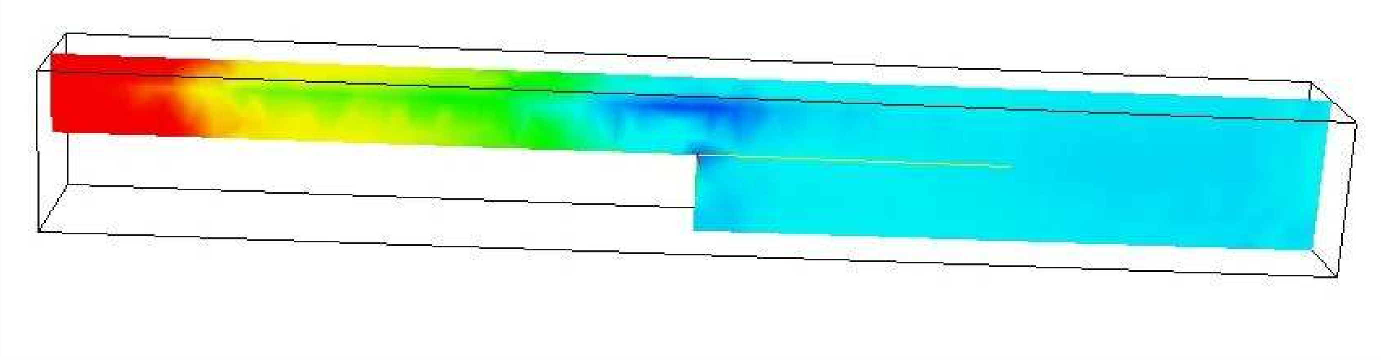
Numerical results: The step test (Re=1000)





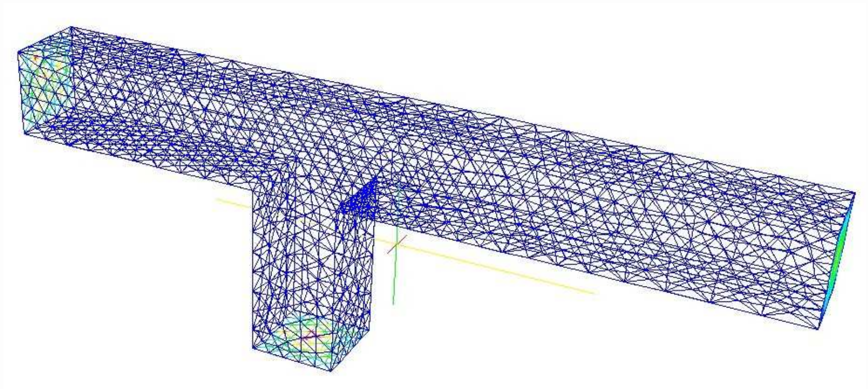
Vorticity lines

Numerical results: The step test (Re=1000)

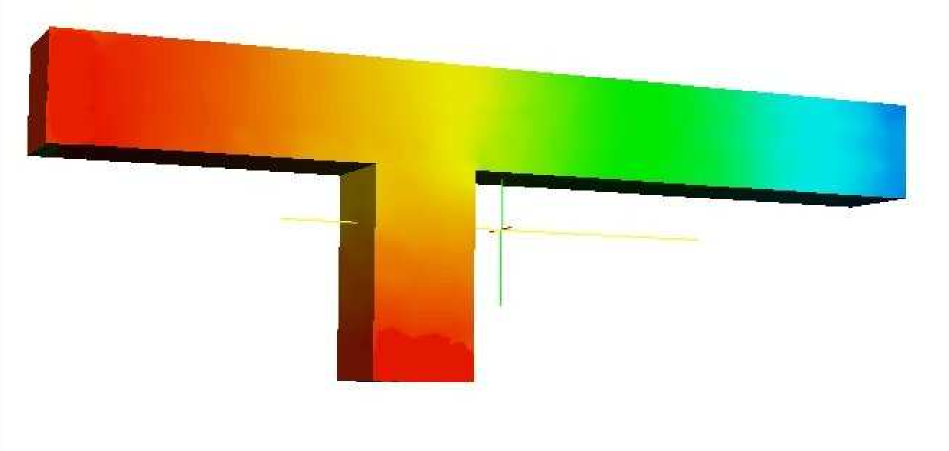


Pressure

Numerical results: T-shaped domain (Re=100)

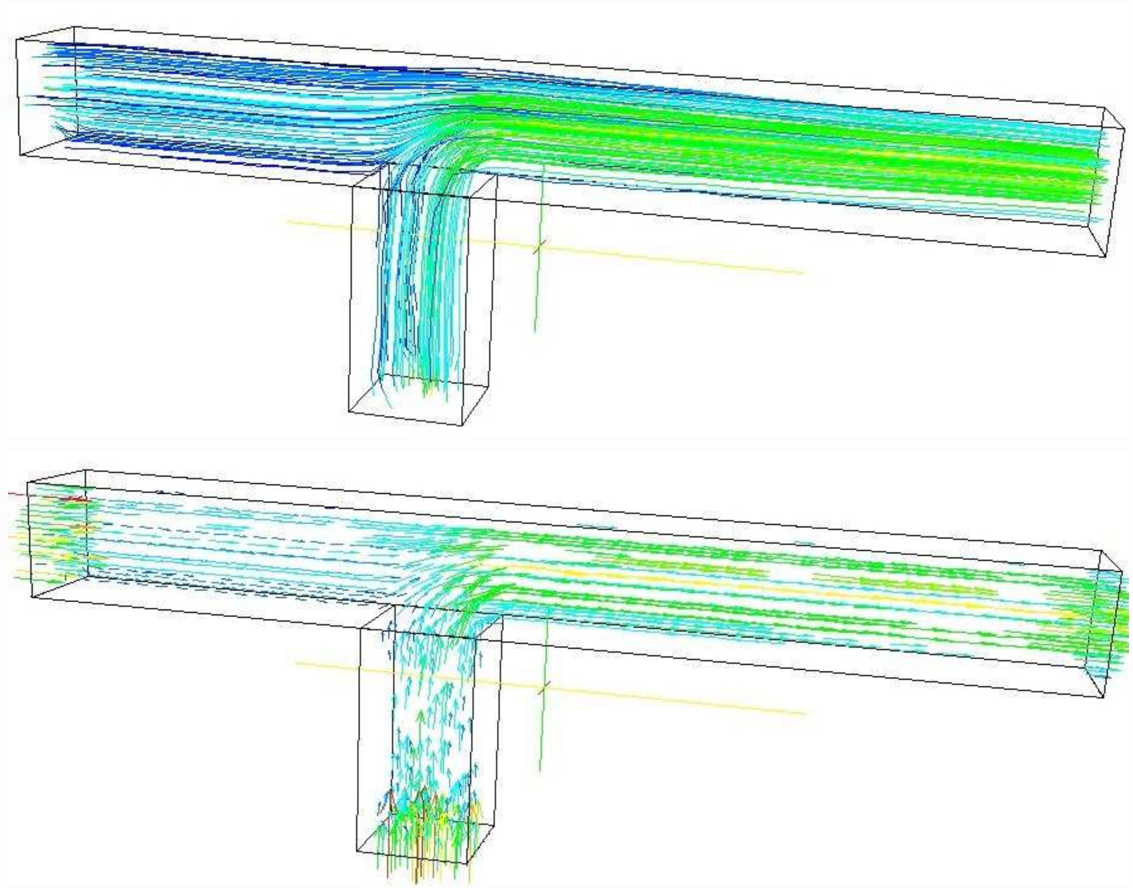


Mesh: 10053 elem., 2469 nodes



Pressure (imposed on inlet and outlet)

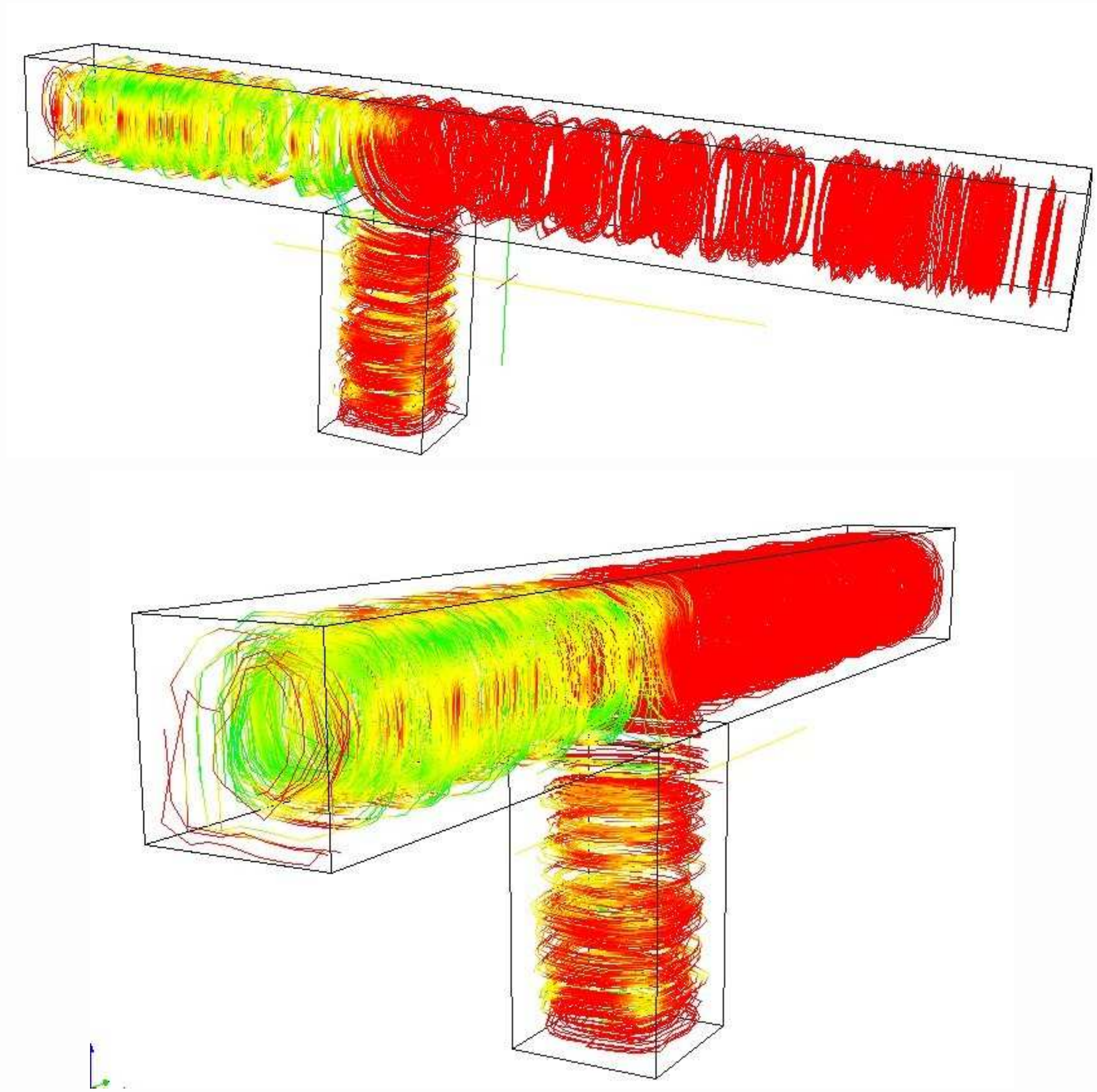
Numerical results: T-shaped domain (Re=100)



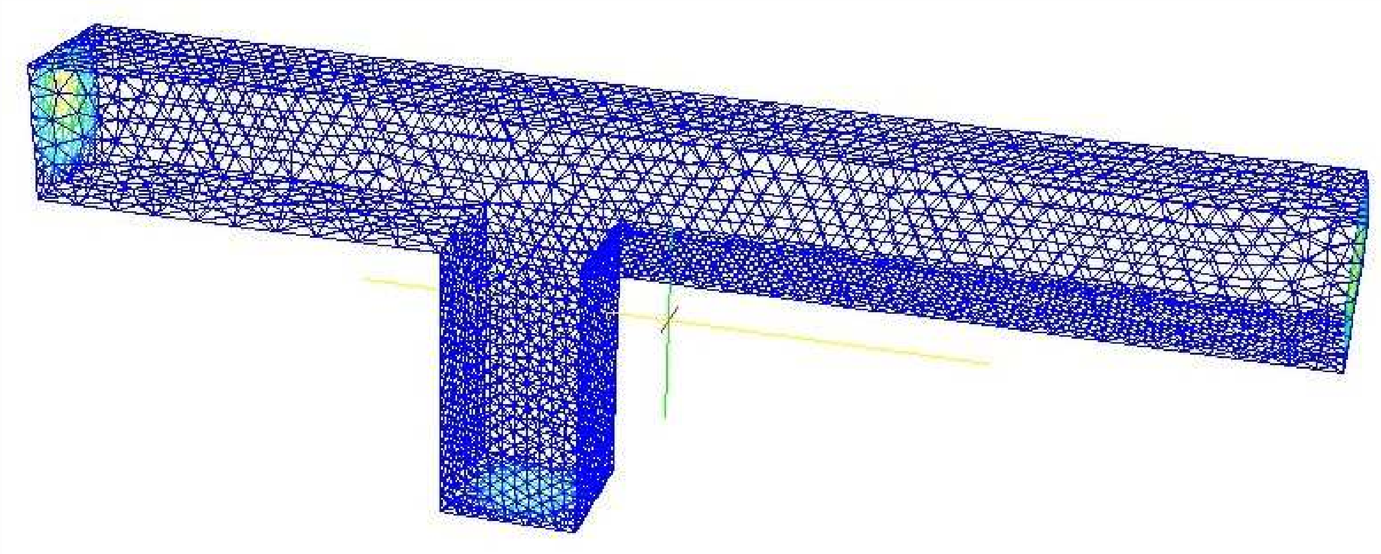
Velocity

Numerical results: T-shaped domain (Re=100)

Vorticity lines



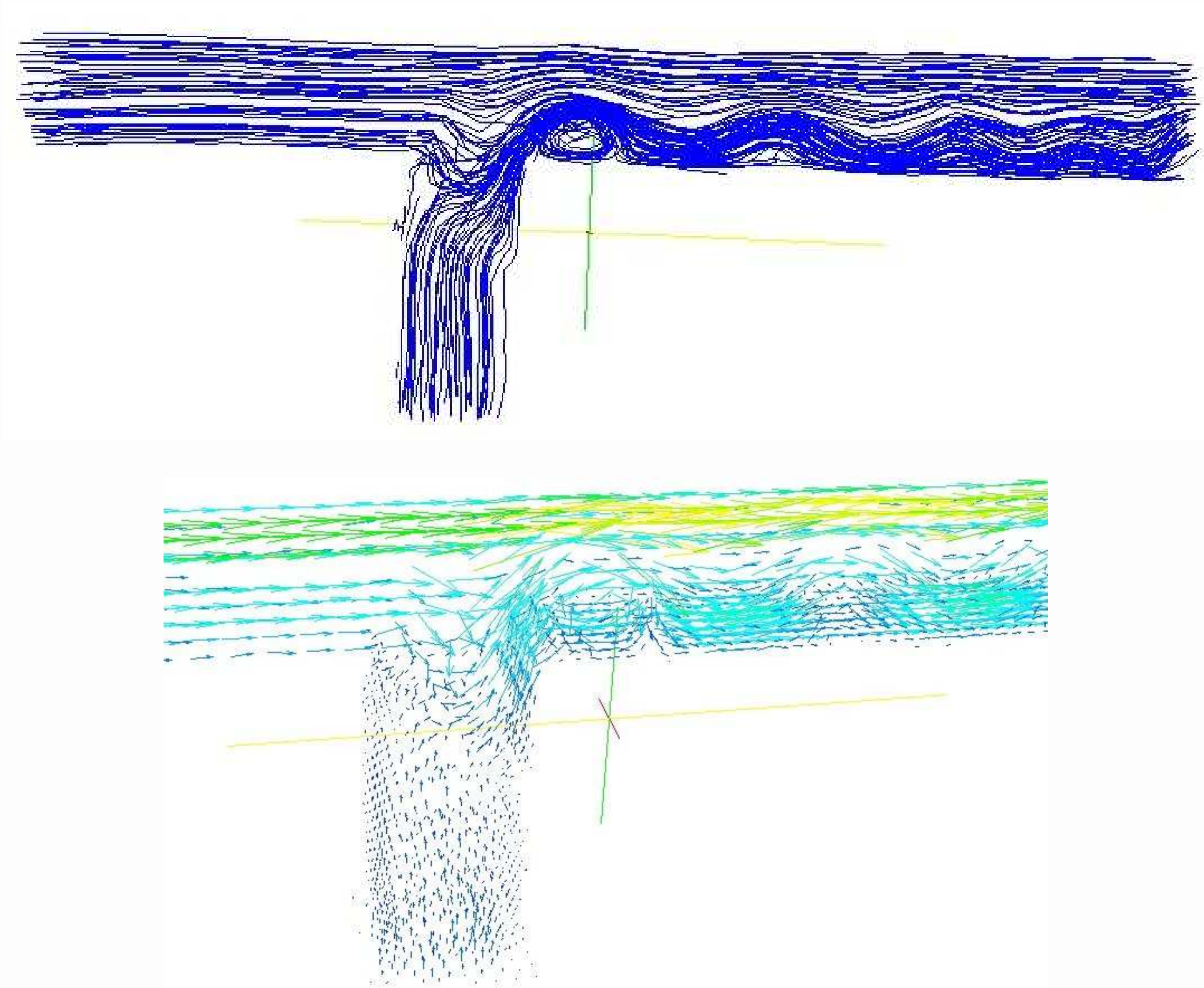
Numerical results: T-shaped domain (Re=10e4)



Mesh: 21985 elem., 5024 nodes

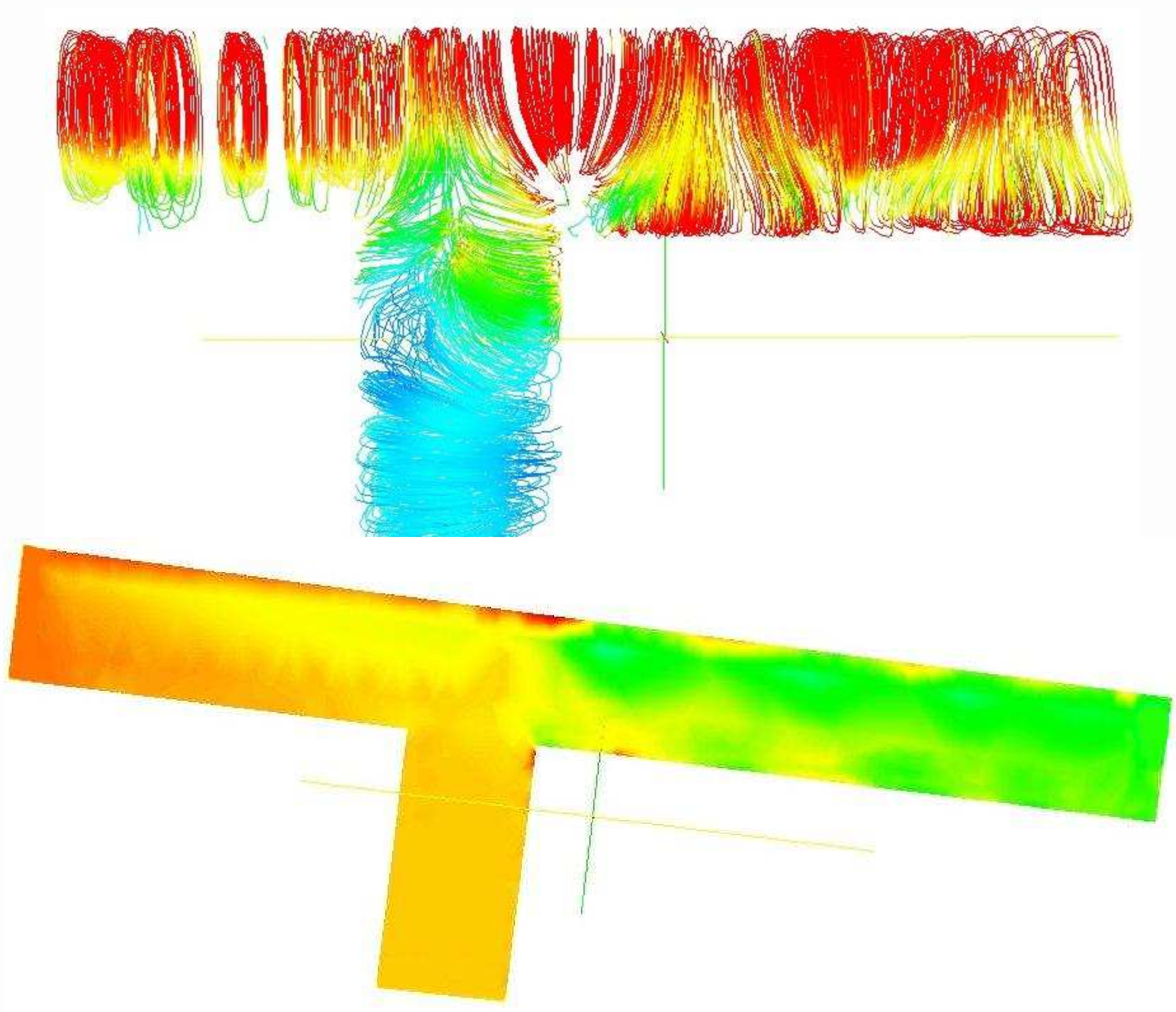
Pressure imposed on the inlet and outlet boundaries.

Numerical results: T-shaped domain (Re=10e4)



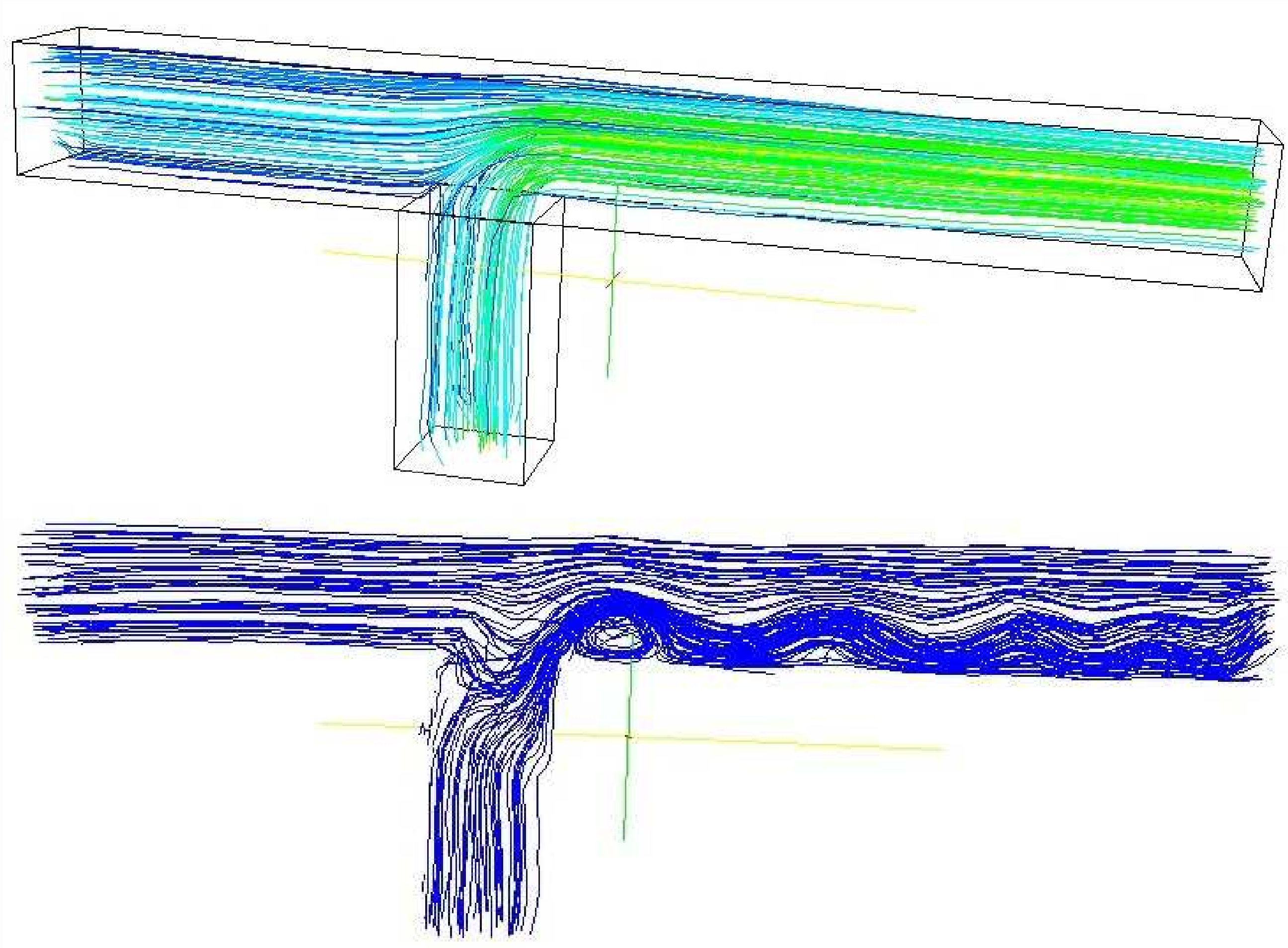
Streamlines and Velocity

Numerical results: T-shaped domain (Re=10e4)



Vorticity lines and pressure

Numerical results: T-shaped domain



Streamlines for Re=100 and Re=10000