

Centered finite volume schemes for diffusion problems on general grids in anisotropic media

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Introduction

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MultiPoint Flux Approximation

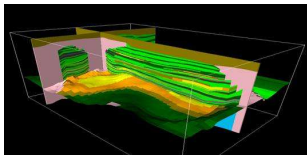
Convergence

Numerical examples

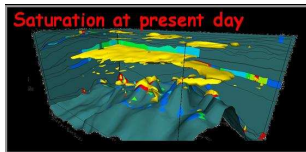
Anisotropic
test case

Anisotropic
heterogeneous
test case

- ▶ Basin and reservoir modeling are used by some oil companies for explorations purposes to reduce exploration risk and cost.
- ▶ Basin simulations :
 - ▶ To improve the present time description of the basin simulating its geological evolution : sedimentation, compaction, erosion; temperature, pressure, HC generation, migration



- ▶ To predict the volume, the location and the compositions of the oil.



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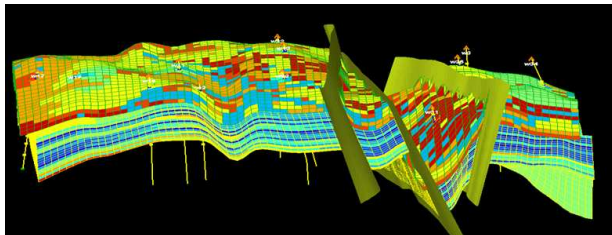
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- ▶ Reservoir simulations :
 - ▶ minimize risk for the development wells
 - ▶ To optimize the oil recovery processes : well location, injection of water, CO_2 ..., thermal, chemical oil recovery,
 - ▶ Prediction of the production history of the reservoir.



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Environmental impact of the oil : production of CO_2 .
 CO_2 geological storage simulation allows :

- ▶ To optimize CO_2 injection processes
- ▶ Predict and manage the risks of CO_2 leakage

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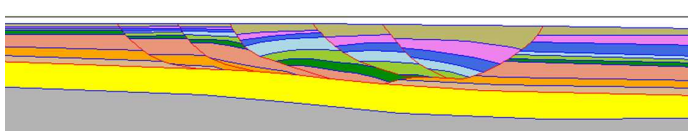
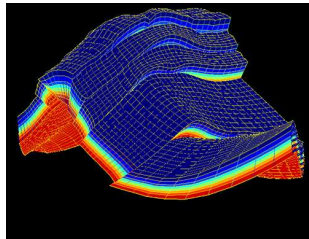
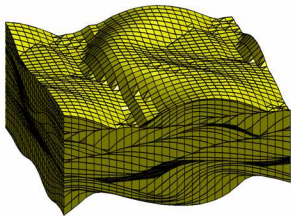
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- ▶ An efficient numerical simulation requires to design accurate and robust discretization schemes for the diffusion fluxes $-\Lambda \nabla P$
- ▶ The mesh has to accurately describe the geological complexity of the heterogeneous porous medium: layers, faults, fractures, channels



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- ▶ Provide a framework to analyse **FV methods in flux formulation**
- ▶ Inspired by [Eymard et al., 2007, Eymard et al., 2008]
- ▶ See also [Droniou, 2002, Di Pietro and Ern, 2008, Agélas et al., 2008]



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- We consider the following problem:

$$\begin{cases} -\operatorname{div}(\Lambda \nabla \bar{u}) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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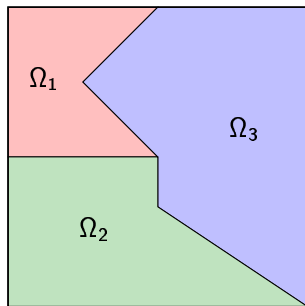
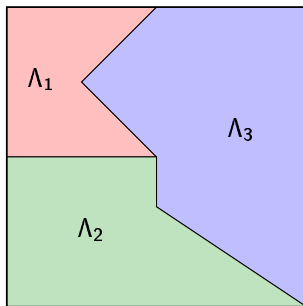
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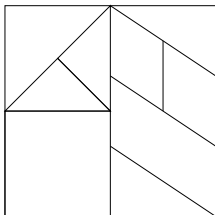
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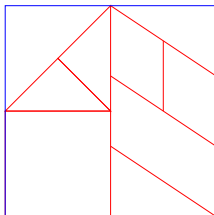
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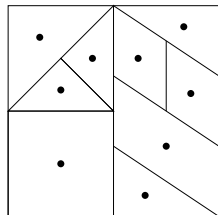
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(a) \mathcal{T}_h



(b) \mathcal{E}_h



(c) \mathcal{P}_h

Figure: Discretization $\mathcal{D}_h = (\mathcal{T}_h, \mathcal{E}_h, \mathcal{P}_h)$ with h the meshsize.

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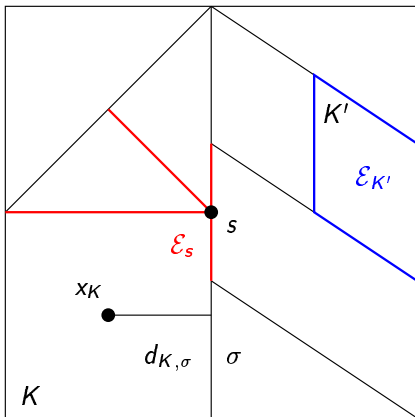
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Green's formula used on each cell K , gives :

$$-\sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} \Lambda \nabla \bar{u} \cdot n_{K,\sigma} = \int_K f$$

Our goal is to find $F_{K,\sigma}$, an approximation of $\int_{\sigma} \Lambda \nabla \bar{u} \cdot n_{K,\sigma}$ such that

$$\begin{aligned} -\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} &= \int_K f, \\ F_{K,\sigma} + F_{L,\sigma} &= 0 \quad K \text{ and } L \text{ sharing } \sigma \end{aligned}$$

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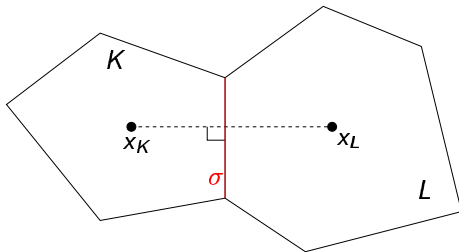
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For $\Lambda = \text{identity matrix}$,
a possible approximation of $\int_{\sigma} \Lambda \nabla \bar{u} \cdot n_{K,\sigma}$ on “orthogonal mesh” is:

$$F_{k,\sigma} = m_{\sigma} \frac{\bar{u}(x_L) - \bar{u}(x_K)}{d(x_K, x_L)}$$



Problem : how to approximate $\int_{\sigma} \Lambda \nabla \bar{u} \cdot n_{K,\sigma}$ on general meshes for anisotropic discontinuous tensor Λ ?

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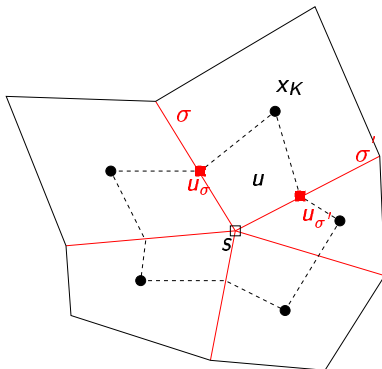
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- ▶ Finite Volume MPFA O method :
The aim : to compute fluxes at half edges around each vertex s .

$$F_{K,\sigma} = m_\sigma \langle \Lambda \rangle_K (\nabla_{\mathcal{D}} u)_K \cdot \mathbf{n}_{K,\sigma}$$



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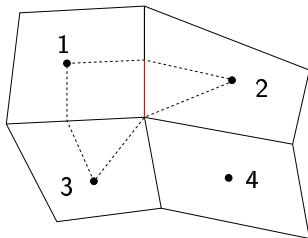
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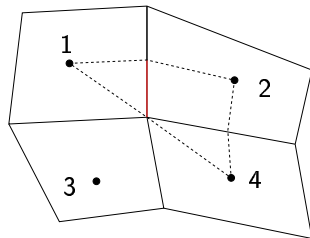
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► Finite Volume MPFA L method :



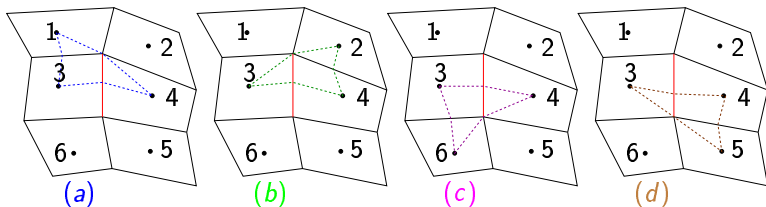
triangle a



triangle b

if $|t_1^a| < |t_2^b|$, *triangle a* is chosen, else *triangle b* is chosen

► Finite Volume MPFA G method :



The final flux $F_{K,\sigma}$ for the cell $K = 3$ and the face σ is :

$$F_{K,\sigma} = \theta_{K,\sigma}^a F_{K,\sigma}^a + \theta_{K,\sigma}^b F_{K,\sigma}^b + \theta_{K,\sigma}^c F_{K,\sigma}^c + \theta_{K,\sigma}^d F_{K,\sigma}^d$$

$$1 = \theta_{K,\sigma}^a + \theta_{K,\sigma}^b + \theta_{K,\sigma}^c + \theta_{K,\sigma}^d$$

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All the previous finite volume schemes provide consistent fluxes if $\Lambda \in C^1$:

$$|F_{K,\sigma}(\varphi_h) - m_\sigma \langle \Lambda \nabla \varphi \rangle_K \cdot \mathbf{n}_{K,\sigma}| \leq C m_\sigma h \text{ for all } \varphi \in C_0^2$$

Question :

- ▶ Do the fluxes remain consistent if Λ is piecewise regular ?

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Yes, if we do not take C_0^2 as the space of test functions but \mathcal{Q} , where \mathcal{Q} is the space of functions $\varphi : \overline{\Omega} \rightarrow \mathbb{R}$ s.t.

- ▶ $\varphi \in C(\overline{\Omega})$, $\varphi = 0$ on $\partial\Omega$ and, for all $i = 1, \dots, N$, $\varphi \in C^2(\overline{\Omega}_i)$;
- ▶ the tangential derivatives of φ are continuous through the interfaces of P_Ω ;
- ▶ the flux of $\nabla\varphi$ directed by Λn is continuous through the interfaces of P_Ω .

Lemma : \mathcal{Q} is dense in $H_0^1(\Omega)$.

Second question : Does convergence hold if the fluxes $F_{K,\sigma}$ are consistent ? **Yes**

The idea is to transpose the spaces and theorems of the continuum case to a discrete case

- ▶ The space H_0^1 becomes $H_{\mathcal{T}_h}(\Omega)$ the space of piecewise constant function on \mathcal{T}_h equipped with a **discrete H_0^1 norm**.
- ▶ The discrete H_0^1 norm, $\|\cdot\|_{\mathcal{T}_h}$ must verify
 - ▶ **Discrete Poincare inequality** : $\|u\|_{L^2} \leq C \|u\|_{\mathcal{T}_h}, \forall u \in H_{\mathcal{T}_h}(\Omega)$
 - ▶ **Translation estimate** : $\forall \xi \in \mathbb{R}^3$ and $\forall u \in H_{\mathcal{T}_h}(\Omega)$,
 $\|u(\cdot + \xi) - u\|_{L^1(\mathbb{R}^3)} \leq C |\xi| \|u\|_{\mathcal{T}_h}$
 - ▶ **Discrete weak gradient** : $\forall u \in H_{\mathcal{T}_h}(\Omega), \|\tilde{\nabla}_{\mathcal{D}} u\|_{L^2(\mathbb{R}^3)} \leq C \|u\|_{\mathcal{T}_h}$

where

$$\tilde{\nabla}_{\mathcal{D}} u(x) = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_{\sigma} (\gamma_{\sigma} u - u_K) n_{K,\sigma} \text{ for a.e } x \in K \in \mathcal{T}_h$$

$$\frac{\gamma_{\sigma} u - u_K}{d_{K,\sigma}} + \frac{\gamma_{\sigma} u - u_L}{d_{L,\sigma}} = 0$$

- ▶ An example of **discrete H_0^1 norm** (see [Eymard et al., 2007, Eymard et al., 2008])

$$\|v\|_{\mathcal{T}_h} \stackrel{\text{def}}{=} \left(\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}} |\gamma_\sigma v - v_K|^2 \right)^{1/2}$$

Properties :

- ▶ $\tilde{\nabla}$ is **weakly convergent**, i.e., if $\{v_h\}_{h \in \mathcal{H}} \rightarrow v \in L^2(\Omega)$ as $h \rightarrow 0$ in $L^2(\mathbb{R}^d)$ and $\|\tilde{\nabla} v_h\|_{[L^2(\mathbb{R}^d)]^d}$ bounded, then

$$\lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \tilde{\nabla} v_h \cdot \Phi = - \int_{\mathbb{R}^d} v \nabla \cdot \Phi, \quad \forall \Phi \in \mathcal{Q}.$$

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- ▶ And the weak formulation of our continuous problem : Find $u \in H_0^1(\Omega)$ s.t., for all $v \in H_0^1(\Omega)$,

$$a(u, v) \stackrel{\text{def}}{=} \int_{\Omega} \Lambda(x) \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f v$$

becomes in the discrete case,

- ▶ Discrete weak formulation: Find $u_h \in H_{\mathcal{T}_h}(\Omega)$ s.t., for all $v \in H_{\mathcal{T}_h}(\Omega)$

$$a_{\mathcal{T}_h}(u_h, v) \stackrel{\text{def}}{=} \sum_{K \in \mathcal{T}_h} -v_K \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h) = \int_{\Omega} f v$$

Assumptions

(P1) \mathcal{Q} is a test space dense in $H_0^1(\Omega)$ s.t. $\mathcal{Q} \subset C_0(\bar{\Omega}) \cap C^2(\bar{\Omega}_i)$

(P2) **Coercivity.** There is $0 < \gamma_1 < +\infty$ independent of h s.t.

$$\forall v \in H_{T_h}(\Omega), \quad a_{T_h}(v, v) \geq \gamma_1 \|v\|_{T_h}^2$$

(P3) **Weak consistency** (L^2 consistency). For all $\varphi \in \mathcal{Q}$ with

$$\varphi_h \stackrel{\text{def}}{=} \{\varphi(x_K)\}_{K \in \mathcal{T}_h} \in H_{T_h}(\Omega)$$

$$\lim_{h \rightarrow 0} \left(\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{d_{K,\sigma}}{m_\sigma} |F_{K,\sigma}(\varphi_h) - m_\sigma \langle \Lambda \nabla \varphi \rangle_K \cdot n_{K,\sigma}|^2 \right) = 0$$

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Theorem (Convergence)

Let $\{\mathcal{D}_h\}_{h \in \mathfrak{H}}$ be an admissible family of discretizations.

Then, as $h \rightarrow 0$, $\{u_h\}_{h \in \mathfrak{H}}$, converges to \bar{u} in $L^2(\Omega)$.

Theorem (Convergence)

Let $\{\mathcal{D}_h\}_{h \in \mathfrak{H}}$ be an admissible family of discretizations.
Then, as $h \rightarrow 0$, $\{u_h\}_{h \in \mathfrak{H}}$, converges to \bar{u} in $L^2(\Omega)$.

Remark

Flux conservativity *is not required* to prove convergence.

Theorem (Gradient reconstruction)

Let

$$\forall K \in \mathcal{T}_h, \quad \overline{\nabla}_{\mathcal{D}} v(x)|_K \stackrel{\text{def}}{=} \frac{1}{m_K} \langle \Lambda \rangle_K^{-1} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(v)(x_\sigma - x_K).$$

Theorem (Gradient reconstruction)

Let

$$\forall K \in \mathcal{T}_h, \quad \overline{\nabla}_{\mathcal{D}} v(x)|_K \stackrel{\text{def}}{=} \frac{1}{m_K} \langle \Lambda \rangle_K^{-1} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(v)(x_\sigma - x_K).$$

Assume that, there is C_2 independent of n s.t. $\forall n \in \mathbb{N}$,

$$\forall v \in H_{\mathcal{T}_h}(\Omega), \quad \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{d_{K,\sigma}}{m_\sigma} |F_{K,\sigma}(v)|^2 \leq C_2 \|v\|_{\mathcal{T}_h,1}^2.$$

Then, as $h \rightarrow 0$, $\{\overline{\nabla}_{\mathcal{D}} u_h\}_{h \in \mathcal{S}}$ converges to $\nabla \bar{u}$ in $[L^2(\Omega)]^d$.

- ▶ **Anisotropic** test case

$$\bar{u} = \sin(\pi x) \sin(\pi y), \quad \Lambda = \text{diag}(0.1, 1)$$

- ▶ **Anisotropic heterogeneous** test case

$$\bar{u} = \begin{cases} g(x, y) & \text{if } x \leq \delta, \\ g(x, y) + \frac{\pi b a_1}{a_2} \cos(b\pi\delta)(x - \delta) \sin(c\pi y) & \text{otherwise} \end{cases}$$

with $g(x, y) \stackrel{\text{def}}{=} \sin(b\pi x) \sin(c\pi y)$ and

$$\Lambda = \begin{cases} \text{diag}(a_1, b_1) & \text{if } x \leq \delta, \\ \text{diag}(a_2, b_2) & \text{otherwise,} \end{cases}$$

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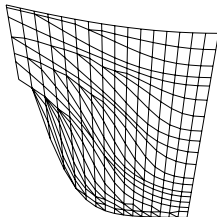
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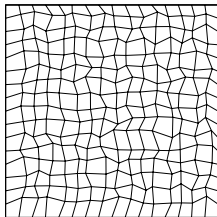
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(a) Basin mesh ($x : y = 10 : 1$)



(b) Randomly perturbed mesh

Figure: Two elements of the mesh families used

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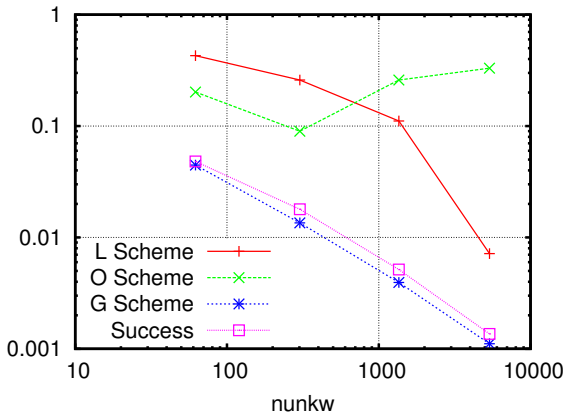


Figure: $\|u_h - \bar{u}\|_{L^2(\Omega)}$ (direct solver)

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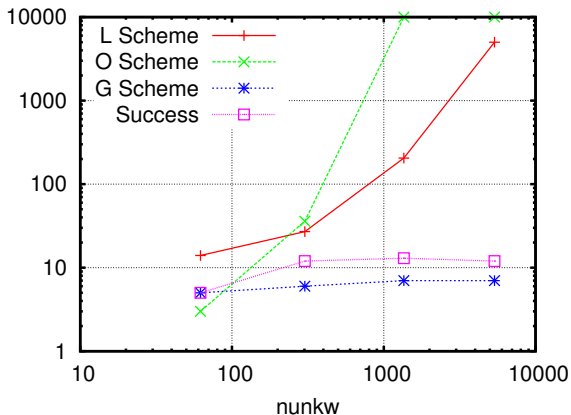


Figure: Number of GMRes+BoomerAMG iterations

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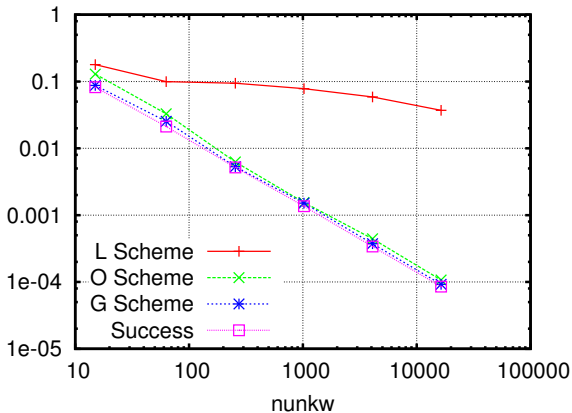


Figure: $\|u_h - \bar{u}\|_{L^2(\Omega)}$ (direct solver)

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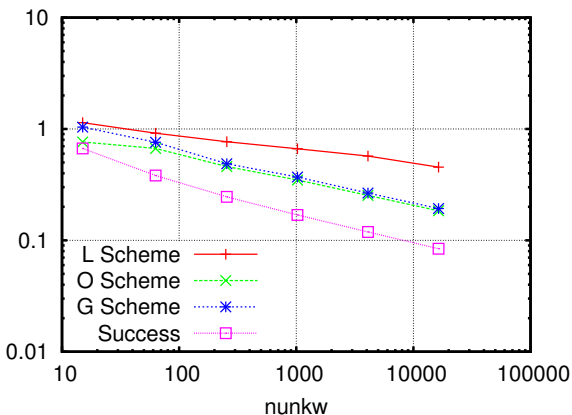


Figure: $\|\bar{\nabla}_{\mathcal{D}} u_h - \nabla \bar{u}\|_{[L^2(\Omega)]^d}$ (direct solver)

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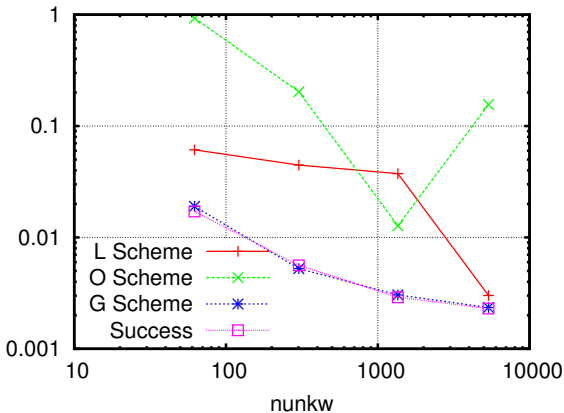


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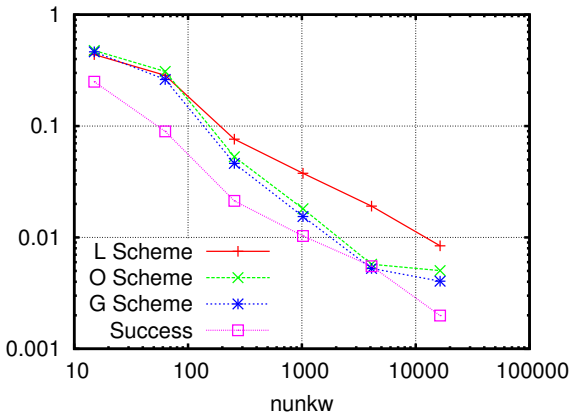


Figure: $\|u_h - \bar{u}\|_{L^2(\Omega)}$ (direct solver)

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