Centered finite volume schemes for diffusion problems on general grids in anisotropic media

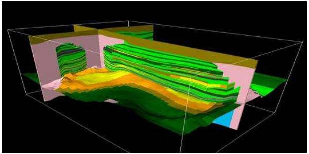
L. Agélas D. A. Di Pietro J. Droniou

IFP (Rueil-Malmaison), IFP (Rueil-Malmaison), Université Montpellier 2 (I3M)

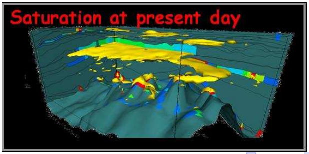
Miniworkshop Momas Méthodes Numériques Pour Les Fluides
Paris, december 4 2008

Introduction

- Basin and reservoir modeling are used by some oil companies for explorations purposes to reduce exploration risk and cost.
- Basin simulations :
  - To improve the present time description of the basin simulating its geological evolution : sedimentation, compaction, erosion; temperature, pressure, HC generation, migration

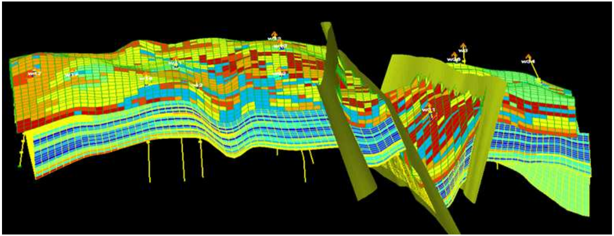


  - To predict the volume, the location and the compositions of the oil.



Introduction

- Reservoir simulations :
  - minimize risk for the development wells
  - To optimize the oil recovery processes : well location, injection of water, CO_2 ..., thermal, chemical oil recovery,
  - Prediction of the production history of the reservoir.



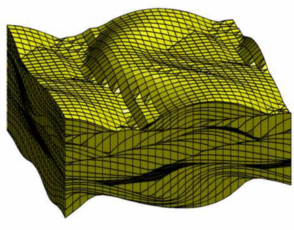
Introduction

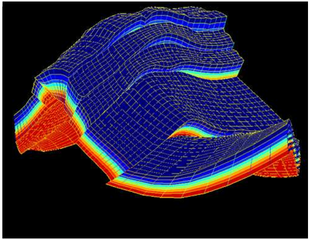
Environmental impact of the oil : production of CO_2.
CO_2 geological storage simulation allows :

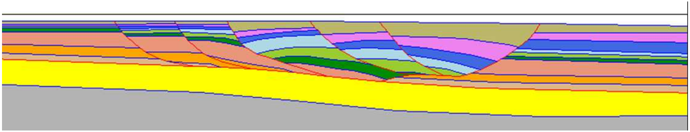
- To optimize CO_2 injection processes
- Predict and manage the risks of CO_2 leakage

Introduction

- An efficient numerical simulation requires to design accurate and robust discretization schemes for the diffusion fluxes $-\Lambda \nabla P$
- The mesh has to accurately describe the geological complexity of the heterogeneous porous medium: layers, faults, fractures, channels







Goals

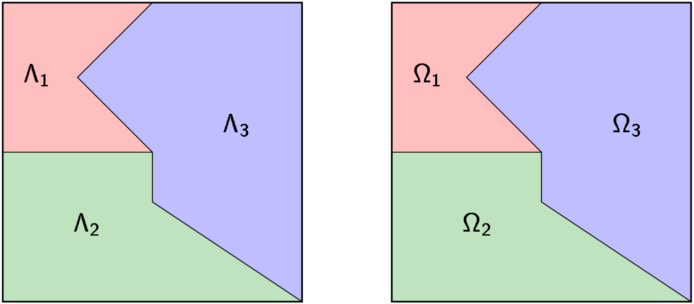
- Provide a framework to analyse FV methods in flux formulation
- Inspired by [Eymard et al., 2007, Eymard et al., 2008]
- See also [Droniou, 2002, Di Pietro and Ern, 2008, Agélas et al., 2008]

Outline

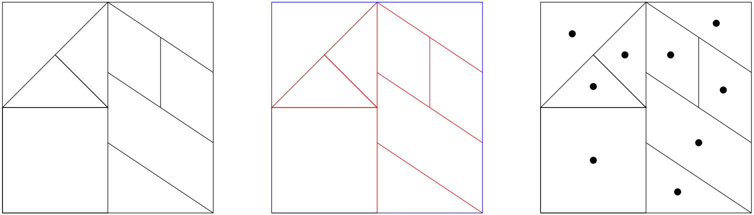
Model problem I

- We consider the following problem:

$$\begin{cases} -\operatorname{div}(\Lambda \nabla \bar{u}) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



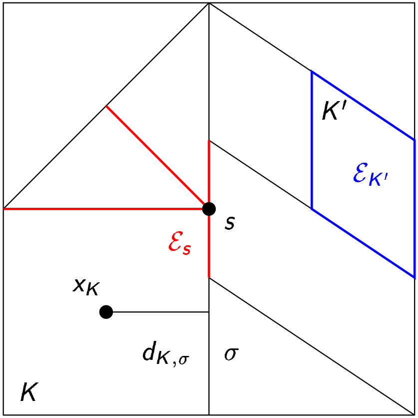
Meshes



(a) $\mathcal{T}_h$ (b) $\mathcal{E}_h$ (c) $\mathcal{P}_h$

Figure: Discretization $\mathcal{D}_h = (\mathcal{T}_h, \mathcal{E}_h, \mathcal{P}_h)$ with h the meshsize.

Notation



Discrete problem I

Green's formula used on each cell K, gives :

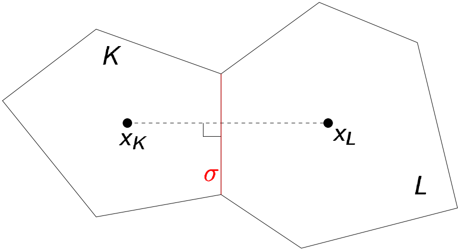
$$-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \Lambda \nabla \bar{u} \cdot \mathrm{n}_{K,\sigma} = \int_K f$$

Our goal is to find $F_{K,\sigma}$, an approximation of $\int_\sigma \Lambda \nabla \bar{u} \cdot \mathrm{n}_{K,\sigma}$ such that

$$\begin{aligned} -\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} &= \int_K f, \\ F_{K,\sigma} + F_{L,\sigma} &= 0 \quad K \text{ and } L \text{ sharing } \sigma \end{aligned}$$

Discrete problem I

For $\Lambda =$ identity matrix ,
a possible approximation of $\int_\sigma \Lambda \nabla \bar{u} \cdot \mathrm{n}_{K,\sigma}$ on "orthogonal mesh" is:
$F_{k,\sigma} = \mathrm{m}_\sigma \frac{\bar{u}(x_L) - \bar{u}(x_K)}{d(x_K, x_L)}$

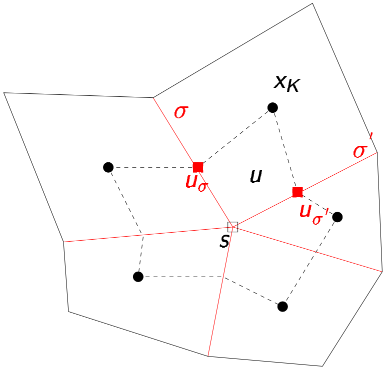


Problem : how to approximate $\int_\sigma \Lambda \nabla \bar{u} \cdot \mathrm{n}_{K,\sigma}$ on general meshes for anisotropic discontinuous tensor Λ ?

MultiPoint Flux Approximation I

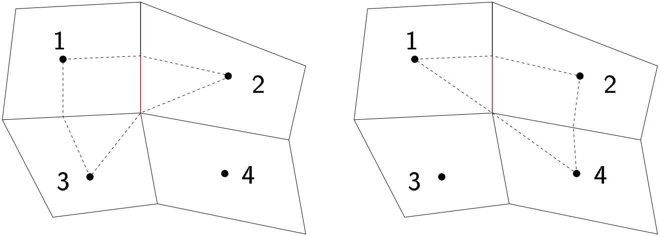
- Finite Volume MPFA O method :
  The aim : to compute fluxes at half edges around each vertex s.

$$F_{K,\sigma} = \mathrm{m}_\sigma \langle\Lambda\rangle_K (\nabla_{\mathcal{D}} u)_K \cdot \mathrm{n}_{K,\sigma}$$



MultiPoint Flux Approximation I

- Finite Volume MPFA L method :

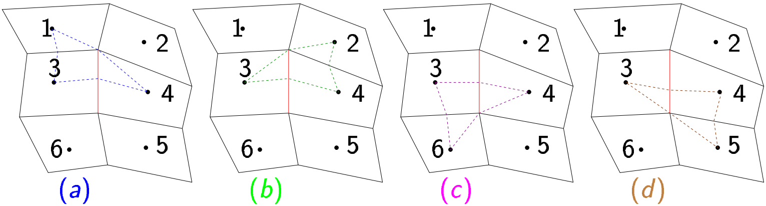


triangle a *triangle b*

if $|t_1^a| < |t_2^b|$, *triangle a* is chosen, else *triangle b* is chosen

MultiPoint Flux Approximation I

- Finite Volume MPFA G method :



The final flux $F_{K,\sigma}$ for the cell $K = 3$ and the face σ is :

$$F_{K,\sigma} = \theta^a_{K,\sigma} F^a_{K,\sigma} + \theta^b_{K,\sigma} F^b_{K,\sigma} + \theta^c_{K,\sigma} F^c_{K,\sigma} + \theta^d_{K,\sigma} F^d_{K,\sigma}$$

$$1 = \theta^a_{K,\sigma} + \theta^b_{K,\sigma} + \theta^c_{K,\sigma} + \theta^d_{K,\sigma}$$

MultiPoint Flux Approximation I

All the previous finite volume schemes provide consistent fluxes if $\Lambda \in C^1$:

$$|F_{K,\sigma}(\varphi_h) - \mathrm{m}_\sigma \langle \Lambda \nabla \varphi \rangle_K \cdot \mathrm{n}_{K,\sigma}| \leq C \ \mathrm{m}_\sigma h \ \text{ for all } \varphi \in C_0^2$$

Question :

- Do the fluxes remain consistent if Λ is piecewise regular ?

MultiPoint Flux Approximation I

Yes, if we do not take C_0^2 as the space of test functions but $\mathcal{Q}$, where $\mathcal{Q}$ is the space of functions $\varphi : \overline{\Omega} \to \mathbb{R}$ s.t.

- $\varphi \in C(\overline{\Omega})$, $\varphi = 0$ on $\partial\Omega$ and, for all $i = 1, \ldots, N$, $\varphi \in C^2(\overline{\Omega_i})$;
- the tangential derivatives of φ are continuous through the interfaces of P_Ω;
- the flux of $\nabla\varphi$ directed by $\Lambda\,\mathrm{n}$ is continuous through the interfaces of P_Ω.

Lemma : $\mathcal{Q}$ is dense in $H_0^1(\Omega)$.
Second question : Does convergence hold if the fluxes $F_{K,\sigma}$, are consistent ? Yes

Convergence I

The idea is to transpose the spaces and theorems of the continuum case to a discrete case

- The space H_0^1 becomes $H_{\mathcal{T}_h}(\Omega)$ the space of piecewise constant function on $\mathcal{T}_h$ equipped with a discrete H_0^1 norm.
- The discrete H_0^1 norm, $\|\cdot\|_{\mathcal{T}_h}$ must verify
  - Discrete Poincare inequality : $\|u\|_{L^2} \leq C\|u\|_{\mathcal{T}_h}, \forall u \in H_{\mathcal{T}_h}(\Omega)$
  - Translation estimate : $\forall \xi \in \mathbb{R}^3$ and $\forall u \in H_{\mathcal{T}_h}(\Omega)$, $\|u(\cdot+\xi) - u\|_{L^1(\mathbb{R}^3)} \leq C\ |\xi|\ \|u\|_{\mathcal{T}_h}$
  - Discrete weak gradient : $\forall u \in H_{\mathcal{T}_h}(\Omega)$, $\|\widetilde{\nabla}_{\mathcal{D}} u\|_{L^2(\mathbb{R}^3)} \leq C\|u\|_{\mathcal{T}_h}$

where

$$\widetilde{\nabla}_{\mathcal{D}} u(x) = \frac{1}{\mathrm{m}_K} \sum_{\sigma \in \mathcal{E}_K} \mathrm{m}_\sigma (\gamma_\sigma u - u_K)\, \mathrm{n}_{K,\sigma} \quad \text{for a.e } x \in K \in \mathcal{T}_h$$

$$\frac{\gamma_\sigma u - u_K}{d_{K,\sigma}} + \frac{\gamma_\sigma u - u_L}{d_{L,\sigma}} = 0$$

Convergence I

- An example of discrete H_0^1 norm (see [Eymard et al., 2007, Eymard et al., 2008])

$$||v||_{\mathcal{T}_h} \stackrel{\text{def}}{=} \left(\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}_\sigma}{d_{K,\sigma}} |\gamma_\sigma v - v_K|^2 \right)^{1/2}$$

Properties :

- $\widetilde{\nabla}$ is weakly convergent, i.e., if $\{v_h\}_{h \in \mathcal{H}} \to v \in L^2(\Omega)$ as $h \to 0$ in $L^2(\mathbb{R}^d)$ and $||\widetilde{\nabla} v_h||_{[L^2(\mathbb{R}^d)]^d}$ bounded, then

$$\lim_{h \to 0} \int_{\mathbb{R}^d} \widetilde{\nabla} v_h \cdot \Phi = -\int_{\mathbb{R}^d} v \nabla \cdot \Phi, \quad \forall \Phi \in \mathcal{Q}.$$

Convergence I

- And the weak formulation of our continuous problem : Find $u \in H_0^1(\Omega)$ s.t., for all $v \in H_0^1(\Omega)$,

$$a(u,v) \stackrel{\text{def}}{=} \int_\Omega \Lambda(x)\nabla u(x) \cdot \nabla v(x)\; dx = \int_\Omega fv$$

becomes in the discrete case,

- Discrete weak formulation: Find $u_h \in H_{\mathcal{T}_h}(\Omega)$ s.t., for all $v \in H_{\mathcal{T}_h}(\Omega)$

$$a_{\mathcal{T}_h}(u_h, v) \stackrel{\text{def}}{=} \sum_{K \in \mathcal{T}_h} -v_K \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h) = \int_\Omega fv$$

Convergence

Assumptions

(P1) $\mathcal{Q}$ is a test space dense in $H_0^1(\Omega)$ s.t. $\mathcal{Q} \subset C_0(\overline{\Omega}) \cap C^2(\overline{\Omega}_i)$

(P2) Coercivity. There is $0 < \gamma_1 < +\infty$ independent of h s.t.

$$\forall v \in H_{\mathcal{T}_h}(\Omega), \quad a_{\mathcal{T}_h}(v,v) \geq \gamma_1 \|v\|^2_{\mathcal{T}_h}$$

(P3) Weak consistency (L^2 consistency). For all $\varphi \in \mathcal{Q}$ with $\varphi_h \stackrel{\text{def}}{=} \{\varphi(x_K)\}_{K \in \mathcal{T}_h} \in H_{\mathcal{T}_h}(\Omega)$

$$\lim_{h \to 0} \left(\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{d_{K,\sigma}}{\mathrm{m}_\sigma} \left| F_{K,\sigma}(\varphi_h) - \mathrm{m}_\sigma \left\langle \Lambda \nabla \varphi \right\rangle_K \cdot \mathrm{n}_{K,\sigma} \right|^2 \right) = 0$$

Main result

Theorem (Convergence)

Let $\{\mathcal{D}_h\}_{h\in\mathfrak{H}}$ be an admissible family of discretizations. Then, as $h \to 0$, $\{u_h\}_{h\in\mathfrak{H}}$, converges to $\bar{u}$ in $L^2(\Omega)$.

Main result

Theorem (Convergence)

Let $\{\mathcal{D}_h\}_{h\in\mathfrak{H}}$ be an admissible family of discretizations. Then, as $h \to 0$, $\{u_h\}_{h\in\mathfrak{H}}$, converges to $\bar{u}$ in $L^2(\Omega)$.

Remark

Flux conservativity is not required to prove convergence.

Gradient reconstruction

Theorem (Gradient reconstruction)

Let

$$\forall K \in \mathcal{T}_h, \quad \overline{\nabla}_{\mathcal{D}} v(x)_{|K} \stackrel{\text{def}}{=} \frac{1}{\mathrm{m}_K} \langle \Lambda \rangle_K^{-1} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(v)(x_\sigma - x_K).$$

Gradient reconstruction

Theorem (Gradient reconstruction)

Let

$$\forall K \in \mathcal{T}_h, \quad \overline{\nabla}_{\mathcal{D}} v(x)_{|K} \stackrel{\text{def}}{=} \frac{1}{\mathrm{m}_K} \langle \Lambda \rangle_K^{-1} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(v)(x_\sigma - x_K).$$

Assume that, there is C_2 independent of n s.t. $\forall n \in \mathbb{N}$,

$$\forall v \in H_{\mathcal{T}_h}(\Omega), \quad \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{d_{K,\sigma}}{\mathrm{m}_\sigma} |F_{K,\sigma}(v)|^2 \leq C_2 \|v\|^2_{\mathcal{T}_h, I}.$$

Then, as $h \to 0$, $\{\overline{\nabla}_{\mathcal{D}} u_h\}_{h \in \mathfrak{H}}$ converges to $\nabla \bar{u}$ in $[L^2(\Omega)]^d$.

Numerical examples I

- Anisotropic test case

$$\bar{u} = \sin(\pi x)\sin(\pi y), \quad \Lambda = \mathrm{diag}(0.1, 1)$$

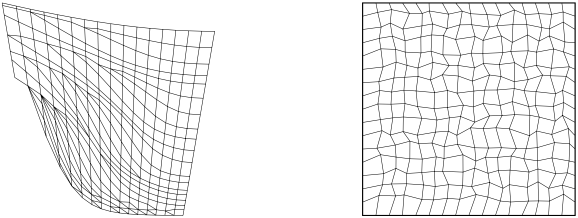
- Anisotropic heterogeneous test case

$$\bar{u} = \begin{cases} g(x,y) & \text{if } x \leq \delta, \\ g(x,y) + \dfrac{\pi b a_1}{a_2}\cos(b\pi\delta)(x-\delta)\sin(c\pi y) & \text{otherwise} \end{cases}$$

with $g(x,y) \stackrel{\text{def}}{=} \sin(b\pi x)\sin(c\pi y)$ and

$$\Lambda = \begin{cases} \mathrm{diag}(a_1, b_1) & \text{if } x \leq \delta, \\ \mathrm{diag}(a_2, b_2) & \text{otherwise,} \end{cases}$$

Numerical examples II



(a) Basin mesh ($x : y = 10 : 1$) (b) Randomly perturbed mesh

Figure: Two elements of the mesh families used

Basin mesh family I

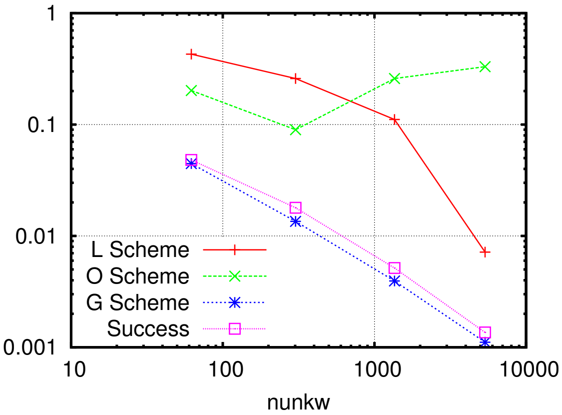


Figure: $\|u_h - \bar{u}\|_{L^2(\Omega)}$ (direct solver)

Basin mesh family II

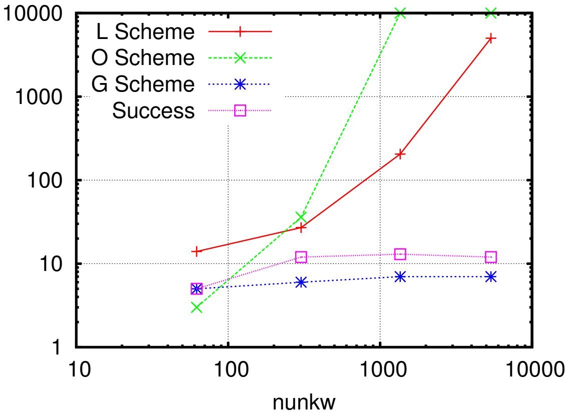


Figure: Number of GMRes+BoomerAMG iterations

Random mesh family I

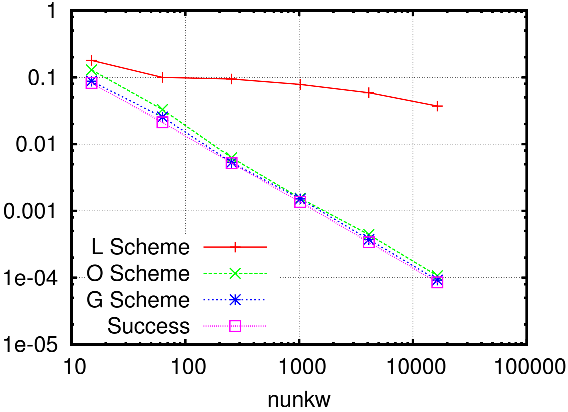


Figure: $\|u_h - \bar{u}\|_{L^2(\Omega)}$ (direct solver)

Random mesh family II

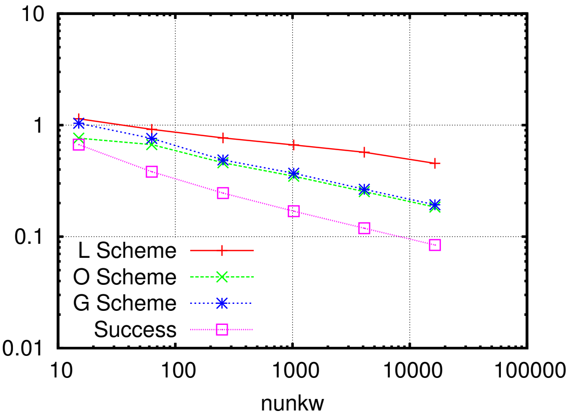


Figure: $\|\overline{\nabla}_{\mathcal{D}} u_h - \nabla \bar{u}\|_{[L^2(\Omega)]^d}$ (direct solver)

Basin mesh family

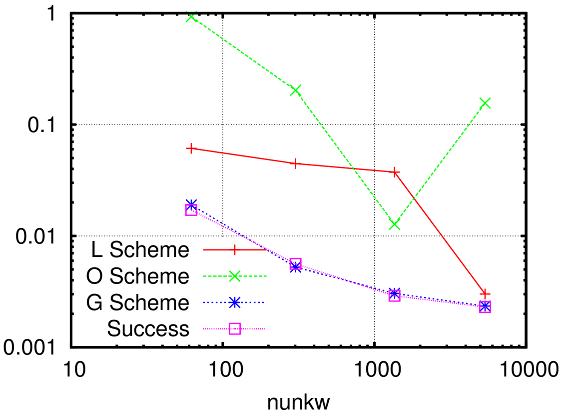


Figure: $\|u_h - \bar{u}\|_{L^2(\Omega)}$ (direct solver)

Random mesh family

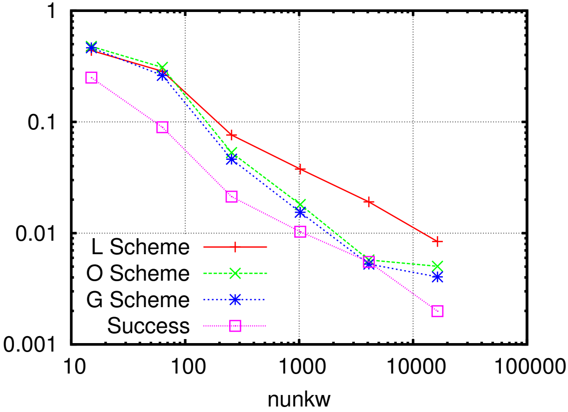


Figure: $\|u_h - \bar{u}\|_{L^2(\Omega)}$ (direct solver)

References

Agélas, L., Di Pietro, D., Eymard, R., and Masson, R. (2008).
An abstract analysis framework for nonconforming approximations of anisotropic heterogeneous diffusion.
Preprint available at http://hal.archives-ouvertes.fr/hal-00318390/fr.
Submitted.

Di Pietro, D. and Ern, A. (2008).
Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations.
Preprint available at http://hal.archives-ouvertes.fr/hal-00278925/fr/.
Submitted.

Droniou, J. (2002).
A density result in Sobolev spaces.
J. Math. Pures Appl., 81(7):697–714.

Eymard, R., Gallouët, T., and Herbin, R. (2007).
A new finite volume scheme for anisotropic diffusion problems on general grids: convergence analysis.
C. R. Math. Acad. Sci., 344(6):3–10.

Eymard, R., Gallouët, T., and Herbin, R. (2008).
Discretization of heterogeneous and anisotropic diffusion problems on general non-conforming meshes. sushi: a scheme using stabilization and hybrid interfaces.
Preprint available at http://hal.archives-ouvertes.fr/.
Submitted.