Schéma éléments finis mixtes-volumes finis pour un modèle d'écoulements eau-gaz en milieux poreux

B. Amaziane, M. Dymitrowska, **M. El Ossmani**, et C. Serres

Journée Scientifique du GdR MoMaS
9 Octobre, 2006 Pau

PLAN

- MODÈLE MATHÉMATIQUE
- DISCRÉTISATION EFMH
- DISCRÉTISATION VF
- STABILITÉ L^∞ ET ESTIMATIONS BV
- RÉSULTATS DE CONVERGENCE
- RÉSULTATS NUMÉRIQUES
- CONCLUSION ET PERSPECTIVES

MODELE MATHEMATIQUE

On considère un modèle d'écoulement diphasique eau-gaz :

$$
(S) \quad \begin{cases}
\text{Trouver } (S_w, S_g, P_w, P_g, \vec{q}_w, \vec{q}_g) \in \Omega \times]0, \tau] \text{tq} : \\
\dfrac{\partial}{\partial t}(\phi \rho_w S_w) + \text{div}(\rho_w \vec{q}_w) = 0, \quad (i) \\
\dfrac{\partial}{\partial t}(\phi \rho_g S_g) + \text{div}(\rho_g \vec{q}_g) = \rho_g Q_g, \quad (ii) \\
\vec{q}_w = -\dfrac{k_{r,w}}{\mu_w} \mathbf{K}(\mathbf{x}) (\nabla P_w - \rho_w \vec{g}), \\
\vec{q}_g = -\dfrac{k_{r,g}}{\mu_g} \mathbf{K}(\mathbf{x}) (\nabla P_g - \rho_g \vec{g}), \\
P_g - P_w = P_c(S_w), \\
S_w + S_g = 1,
\end{cases}
$$

- la loi de van Genuchten

$$
P_c(S_w) = \frac{1}{\beta}\Big(\frac{1}{S_{w,e}^{\frac{1}{m}}} - 1\Big)^{\frac{1}{n}}, \; k_{r,w} = \sqrt{S_{we}}\left[1 - \left(1 - S_{we}^{1/m}\right)^m\right]^2,
$$

$$
k_{r,g} = (1 - S_{we})^{1/p}\left[1 - S_{we}^{1/m}\right]^{2m}.
$$

- la mobilité de la phase $\alpha = w, g$

$$\lambda_\alpha = \lambda_\alpha(S) = \frac{k_{r,\alpha}}{\mu_\alpha}$$

- la mobilité totale $\quad \lambda = \lambda_w + \lambda_g$
- la moyenne harmonique des mobilités

$$\bar{\lambda} = \bar{\lambda}(S) = \frac{\lambda_w(S)\lambda_g(S)}{\lambda_w(S) + \lambda_g(S)}$$

- la fonction du flux fractionnaire

$$f_\alpha = f_\alpha(S) = \frac{\lambda_\alpha}{\lambda}$$

- la vitesse totale $\quad \vec{q} = \vec{q}_w + \vec{q}_g$
- la pression globale $\quad P = \frac{P_w + P_g}{2} + \gamma(S)$

$$\gamma(S) = \frac{P_c(S)}{2} + \int_S^1 f_w P_c^{'}(\xi)\, d\xi,$$

$$\lambda \nabla P = \lambda_w \nabla P_w + \lambda_g \nabla P_g$$

$$\vec{q} = -K(x)\lambda(\nabla P - \bar{\rho}\vec{g}),\ \bar{\rho} = \frac{\lambda_w \rho_w + \lambda_g \rho_g}{\lambda}$$

$$\frac{(i)}{\rho_w} + \frac{(ii)}{\rho_g} : \mathrm{div}\ \vec{q} = -\frac{\partial \phi}{\partial t} - \sum_{\alpha=w}^{\alpha=g} \frac{1}{\rho_\alpha}\left(\phi S_\alpha \frac{\partial \rho_\alpha}{\partial t} + \vec{q}_\alpha . \nabla \rho_\alpha\right) + Q_g .$$

$$\frac{1}{\rho_g}\frac{\partial \rho_g}{\partial t} = \beta_g \left(\frac{\partial P}{\partial t} + (1 - f_g)\frac{\partial P_c}{\partial t}\right)$$

$$\frac{1}{\rho_w}\frac{\partial \rho_w}{\partial t} = \beta_w \left(\frac{\partial P}{\partial t} - (1 - f_w)\frac{\partial P_c}{\partial t}\right)$$

$$\frac{1}{\rho_g}\nabla \rho_g = \beta_g \left(\nabla P + (1 - f_g)\nabla P_c\right),$$

$$\frac{1}{\rho_w}\nabla \rho_w = \beta_w \left(\nabla P - (1 - f_w)\nabla P_c\right),$$

$$\beta_\alpha = \frac{1}{\rho_\alpha}\frac{\partial \rho_\alpha}{\partial P_\alpha}.$$

MODELE MATHEMATIQUE...

$$\mathrm{div}\ \vec{q} = -\frac{\partial \phi}{\partial t} - \phi(S_w\beta_w + S_g\beta_g)\frac{\partial P}{\partial t} - \phi(S_g\beta_g f_w - S_w\beta_w f_g)\frac{\partial P_c}{\partial t}$$
$$+ (\beta_g\vec{q}_g + \beta_w\vec{q}_w).\nabla P + (\beta_g f_w\vec{q}_g - \beta_w f_g\vec{q}_w)\nabla P_c + Q_g,$$

ou encore

$$\mathrm{div}\ \vec{q} + \phi(S_w\beta_w + S_g\beta_g)\frac{\partial P}{\partial t} = -\frac{\partial \phi}{\partial t} + Q_g - \cancel{\phi(S_g\beta_g f_w - S_w\beta_w f_g)\frac{\partial P_c}{\partial t}}$$
$$\cancel{(\beta_g f_w\vec{q}_g - \beta_w f_g\vec{q}_w).\nabla P_c}$$
$$\cancel{(\beta_g\vec{q}_g + \beta_w\vec{q}_w).\nabla P}.$$

MODELE MATHEMATIQUE ...

Si on suppose que $\rho_w = Cte$ (l'eau est incompressible) et que $\rho_g = \sigma_g P_g$ (gaz parfait)

- Pression globale

$$\begin{cases} \vec{q} = -\mathbf{K}(\mathbf{x})\Lambda(S)\ (\nabla P - \overline{\rho}\vec{g}), & \text{dans } \Omega\times]0,\tau] \\ \beta(P,S)\dfrac{\partial P}{\partial t} + \mathrm{div}\ \vec{q} = Q_g & \text{dans } \Omega\times]0,\tau] \end{cases}$$

$$\beta(P,S) = \phi\frac{1-S}{P-\eta(S)} + \frac{\partial \phi}{\partial P}$$

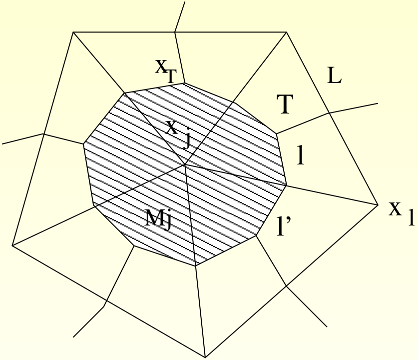
$\eta(S) = \int_S^1 f_w(\xi)P_c'(\xi)d\xi$

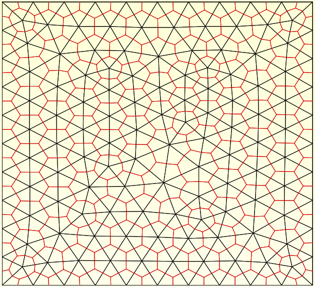
- Saturation en eau

$$\frac{\partial}{\partial t}(\phi\rho_w S) + \mathrm{div}\left[\rho_w f_w \vec{q} + \rho_w\overline{\lambda}(S)\mathbf{K}(\mathbf{x})(\rho_w - \rho_g)\vec{g}\right]$$
$$-\,\mathrm{div}\,(\rho_w \mathbf{K}(\mathbf{x})\nabla\alpha(S)) = 0 \qquad \text{dans } \Omega\times]0,\tau]$$

$\alpha(S) = -\int_0^S \overline{\lambda}(\xi)P_c'(\xi)\,d\xi$

SCHÉMA IMPES EFMH-VF





Maillage dual de Voronoï

- $\Lambda_h = (T_i)_{i=0,\ldots,Ne}$ **triangulation admissible de $\overline{\Omega}$.**
- $\Sigma_h = (M_i)_{i=0,\ldots,Ns}$ **Maillage dual de Voronoï.**

DISCRETISATION EFMH

Soit $(\mathcal{T}_h)_{h>0}$ une triangulation régulière de Ω.
$T \in (\mathcal{T}_h)_{h>0}$: triangle, $L \in T$ arête de T. On pose : $\beta = \frac{\beta^n}{\Delta t^n}$,
$P^{n+1} = P$, $\vec{q}^{n+1} = \vec{q}$, $Q = Q_g^{n+1} + P^n \frac{\beta_t^n}{\Delta t^n}$, $A(x) = \Lambda^n K$,
$P_D^{n+1} = P_D$ et $q_N^{n+1} = q_N$, avec $P = P_D / \Gamma_D$ et $\vec{q}.\vec{n} = q_N / \Gamma_N$

$$\begin{cases} \int_{\mathcal{T}} A^{-1} \vec{q}_h \cdot \vec{s}_h \, d\mathcal{T} - \int_{\mathcal{T}} P_h \text{div } \vec{s}_h \, d\mathcal{T} + \sum_{L \in \mathcal{T}} \int_L \lambda_h \vec{s}_h \cdot \vec{n} \, dL = 0 & \forall \vec{s}_h \in RT_0 \\ \int_{\mathcal{T}} \beta P_h w_h \, d\mathcal{T} + \int_{\mathcal{T}} w_h \text{div } \vec{q}_h \, d\mathcal{T} = \int_{\mathcal{T}} Q w_h \, d\mathcal{T} & \forall w_h \in M^h \\ \sum_{\mathcal{T} \in T_h} \sum_{L \in \mathcal{T}} \int_L \mu_h \vec{q}_h \cdot \vec{n} \, dL = \sum_{L \in \Gamma_N} \int_L q_N \mu_h \, dL & \forall \mu_h \in L_0^h \end{cases}$$

RT_0 est l'espace de Raviart-Thomas de plus bas degré.

- **Etape 1 :** Résoudre un système linéaire avec une matrice symétrique définie positive pour λ_h.
- **Etape 2 :** Résoudre un système linéaire 3×3 sur chaque **élément** pour obtenir $\vec{q}_h$ et p_h.

SVF Semi-Implicite

- Approximation explicite pour le terme de Convection : Schéma de Godunov.
- Approximation EF $\mathcal{P}_1$ implicite pour le terme de Diffusion.

SVF Semi-Implicite :

$$S_{M_j^{n+1}} = \frac{\phi_{M_j}^{n}}{\phi_{M_j}^{n+1}} \frac{\rho_{w,M_j}^{n}}{\rho_{w,M_j}^{n+1}} S_{M_j^n} + \frac{\Delta t_n}{\phi_{M_j}^{n+1} |M_j|} \sum_{I \in \partial M_j} (f_w(S_{M_I}^n) - f_w(S_{M_j}^n))(-\vec{q}_I \cdot \vec{n}_{M_j,I})^+ |I|$$

$$+ \frac{\Delta t_n}{\phi_{M_j}^{n+1} |M_j|} \sum_{I \in \partial M_j \backslash \Gamma} \mathbf{K}(\mathbf{x})_I \left(\alpha(S_{M_I}^{n+1}) - \alpha(S_{M_j}^{n+1}) \right) \frac{D_{M_j I}}{\delta_{M_j I}} |I|$$

où $D_{M_j I} = -\frac{|T|}{|I|} \delta_{M_j I} \nabla N_{M_I,T} \cdot K_T \nabla N_{M_j,T}$ avec $N_{\Omega_I,T}$ sont les fonctions de base d'élément fini $\mathcal{P}_1$ et $K_T = \frac{1}{|T|} \int_T \mathbf{K}(\mathbf{x}) dT$.

- Pour cette approximation on a : $0 < D_- \leq D_{M_j I} \leq D^+ < \infty$.
- Conservation de la masse sur chaque élément, principe du maximum discret, hétérogénéités anisotropiques .

Stabilité L^∞ et Estimations BV

- **Sous certaines hypothèses sur les données et la condition** $CFL := \frac{\Delta t_n}{h}\left(C^n_{q,h} \sup_{0\le S\le 1} f'_w(S)\right) \le 1$

$$C^n_{q,h} = \max_{M_j} \sum_{l\in\partial M_j} \frac{h\,|l|\,(-\vec{q}_l\cdot\vec{n}_{M_j,l})^+}{\phi_M^{n+1}\,|M_j|}$$

le schéma est L^∞ stable, de plus,

$$0 \le \mathbf{S}^{\mathbf{n}}_{\mathbf{M_j}} \le 1,\ \forall\, n,j$$

- **Sous certaines hypothèses sur les données et la condition CFL**, nous avons les estimations BV suivantes :

$$\sum_{n,j} \Delta t^n |M_j| \left(\alpha(S^{n+1}_{M_j}) - \alpha(S^n_{M_j})\right)^2 \le C\Delta t,$$
$$\sum_{n,j} \Delta t^n |l| \left(\alpha(S^n_{M_l}) - \alpha(S^n_{M_j})\right)^2 \le Ch,$$

Résultats de Convergence

Théorème

Sous certains hypothèses sur les données et la condition CFL la solution approchée S_h converge vers S dans $L^2(Q)$, $\vec{q}_h$ converge vers $\vec{q}$ et p_h converge vers p qd h et Δt tendent vers zero.

- **Idée de la preuve :** On utilise une formulation faible pour l'équation de saturation et pour l'équation de pression, le processus de convergence est obtenu en utilisant la stabilité L^∞, estimations BV

Résultats Numériques Test : CU

Figure: Domaine de calcul

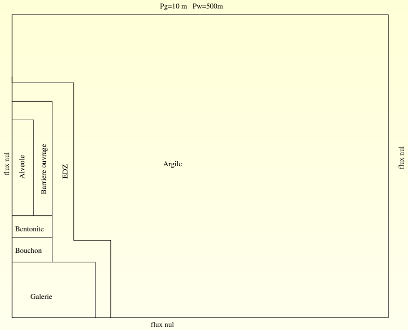


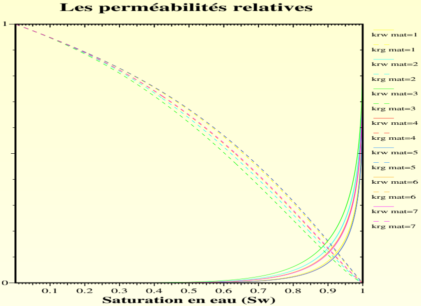
Figure: Conditions initiales et aux bords

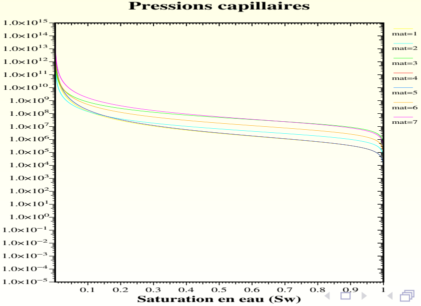
Paramètres physiques retenus

1.000e+00 # Viscosite de l'eau #
1.0000e-02 # Viscosite du gaz #
1.000e+03 # Densite de l'eau #
8.0000e-02 # Densite du gaz #
0.0000e+00 # Coeff. de compressibilites de l'eau #
8.0000e-03 # Coeff. de compressibilites du gaz #
2.3000e-06 # Coeff. de compressibilites du solide #

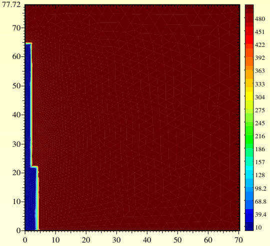
Kxx	Kxy	Kyy	ϕ	P_e	n_{VG}	S_{wr}	S_{gr}	
$5e^{-09}$	0.0	$5e^{-09}$	$3.5e^{-01}$	$6.0e^{+01}$	1.417	0.0	0.0	(galerie)
$1e^{-11}$	0.0	$1e^{-11}$	$1.5e^{-01}$	$2.0e^{+02}$	1.54	0.0	0.0	(bouchon)
$1e^{-13}$	0.0	$1e^{-13}$	$3.5e^{-01}$	$1.8e^{+03}$	1.61	0.0	0.0	(bentonite)
$5e^{-09}$	0.0	$5e^{-09}$	$3.6e^{-01}$	$6.0e^{+01}$	1.40	0.0	0.0	(alveole)
$5e^{-09}$	0.0	$5e^{-09}$	$3.6e^{-01}$	$6.0e^{+01}$	1.40	0.0	0.0	(BO)
$5e^{-11}$	0.0	$5e^{-11}$	$1.6e^{-01}$	$5.0e^{+01}$	1.50	0.0	0.0	(EDZ)
$5e^{-14}$	0.0	$5e^{-14}$	$1.5e^{-01}$	$1.5e^{+03}$	1.49	0.0	0.0	(argile)

Lois physiques

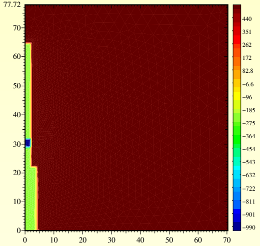




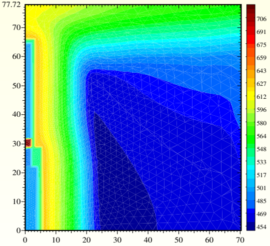
Pressure of gaz (m) at T=0.000000e+00



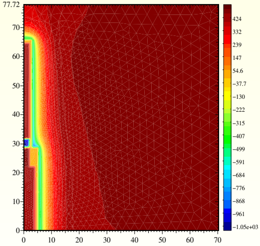
Pressure of water (m) at T=0.000000e+00''years

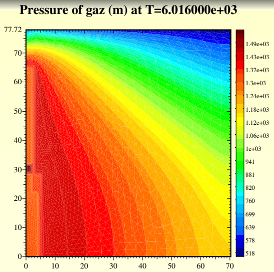


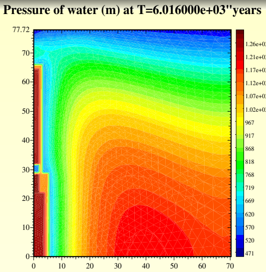
Pressure of gaz (m) at T=1.050413e+03

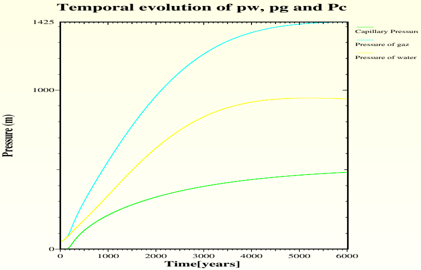


Pressure of water (m) at T=1.050413e+03''years

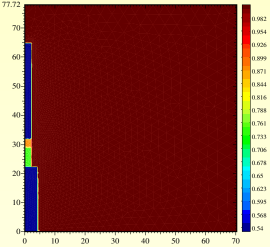



Pressure of gaz (m) at T=6.016000e+03
77.72
1.49e+03
1.43e+03
1.37e+03
1.3e+03
1.24e+03
1.18e+03
1.12e+03
1.06e+03
1e+03
941
881
820
760
699
639
578
518

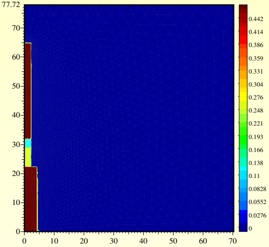

Pressure of water (m) at T=6.016000e+03"years
77.72
1.26e+03
1.21e+03
1.17e+03
1.12e+03
1.07e+03
1.02e+03
967
917
868
818
768
719
669
620
570
520
471


Temporal evolution of pw, pg and Pc
Capillary Pressure
Pressure of gaz
Pressure of water
Pressure (m)
Time[years]

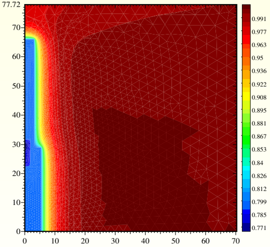
Saturation of water at t=0.0 years



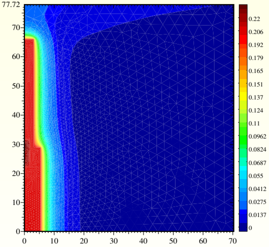
Saturation of gaz at t=0.0 years



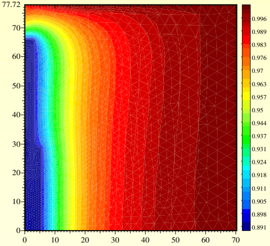
Saturation of water about 1000 years



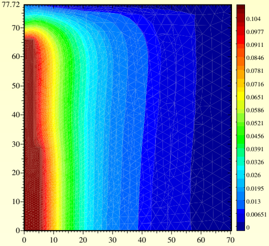
Saturation of gaz about 1000 years



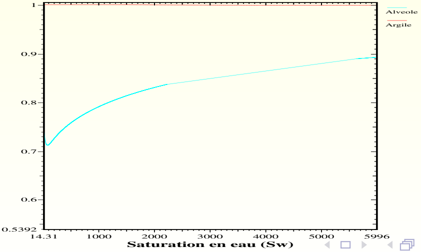
Saturation of water about 6000 years



Saturation of gaz about 6000 years



Temporal evolution of water saturation



Résultats Numériques Test : Couplex-gaz

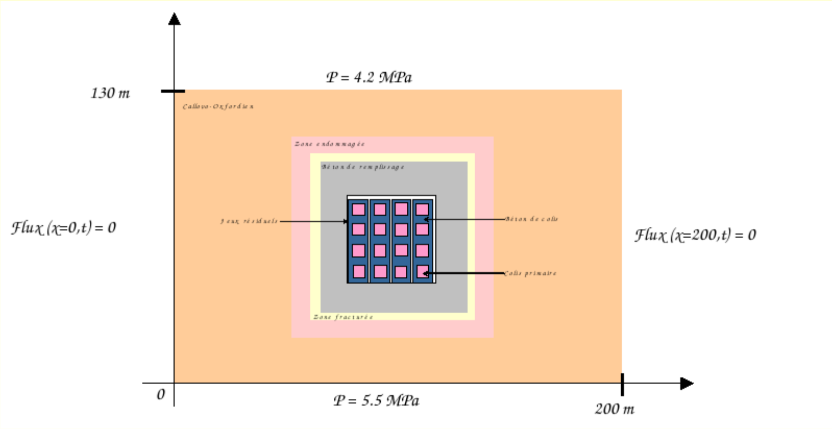


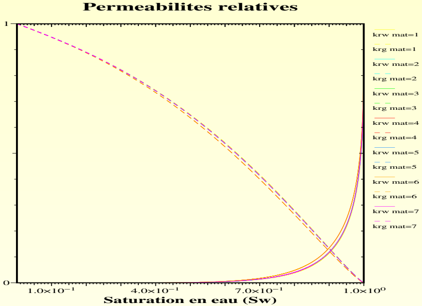
Figure: Conditions initiales et aux bords

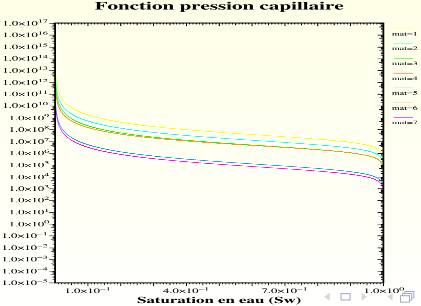
Paramètres physiques retenus

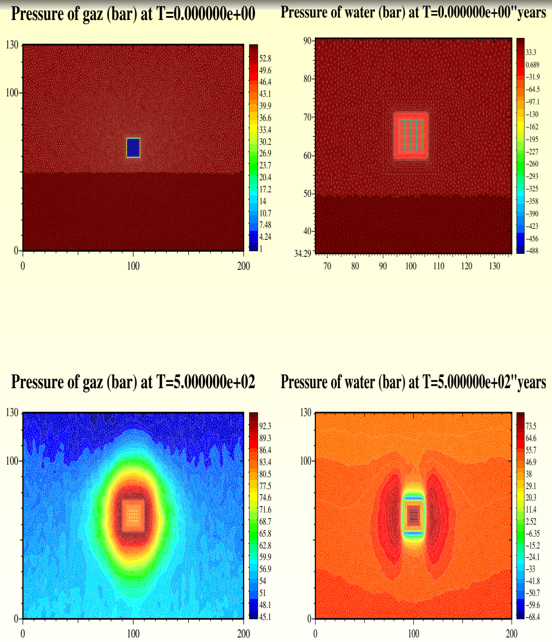
7.9800e-04	# Viscosite de l'eau #
9.0000e-06	# Viscosite du gaz #
9.9571e+02	# Densite de l'eau #
3.16000e-07	# Densite du gaz #
4.5000e-10	# Coeff. de compressibilites de l'eau #
1.0000e-07	# Coeff. de compressibilites du gaz #
4.5000e-10	# Coeff. de compressibilites du solide #

Kxx	Kxy	Kyy	ϕ	P_e	n_{VG}	S_{wr}	S_{gr}	
$5e^{-21}$	0.0	$5e^{-20}$	$1.5e^{-01}$	$15.0e^{+06}$	1.5	0.4	0.0	(COX)
$1e^{-18}$	0.0	$1e^{-18}$	$1.5e^{-01}$	$5.0e^{+06}$	1.5	0.2	0.0	(EDZE)
$1e^{-16}$	0.0	$1e^{-16}$	$1.6e^{-01}$	$2.0e^{+06}$	1.5	0.1	0.0	(EDZR)
$1e^{-18}$	0.0	$1e^{-18}$	$3.0e^{-01}$	$2.0e^{+06}$	1.54	0.01	0.0	(Béton)
$1e^{-12}$	0.0	$1e^{-12}$	$1.0e^{+00}$	$5.0e^{+04}$	1.50	0.01	0.0	(Jeux)
$1e^{-19}$	0.0	$1e^{-19}$	$1.5e^{-01}$	$2.0e^{+06}$	1.54	0.01	0.0	(Béton de colis)
$1e^{-15}$	0.0	$1e^{-15}$	$2.5e^{-01}$	$3.0e^{+04}$	1.5	0.01	0.0	(Alveole)

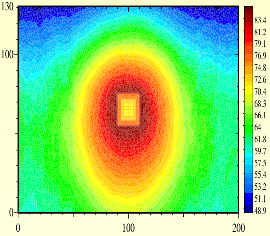
Lois physiques



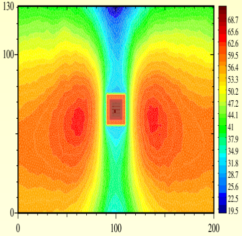



Pressure of gaz (bar) at T=0.000000e+00
130
100
0
0
100
200
52.8
49.6
46.4
43.1
39.9
36.6
33.4
30.2
26.9
23.7
20.4
17.2
14
10.7
7.48
4.24
1
Pressure of water (bar) at T=0.000000e+00"years
90
80
70
60
50
40
34.29
70
80
90
100
110
120
130
33.3
0.689
-31.9
-64.5
-97.1
-130
-162
-195
-227
-260
-293
-325
-358
-390
-423
-456
-488
Pressure of gaz (bar) at T=5.000000e+02
130
100
0
0
100
200
92.3
89.3
86.4
83.4
80.5
77.5
74.6
71.6
68.7
65.8
62.8
59.9
56.9
54
51
48.1
45.1
Pressure of water (bar) at T=5.000000e+02"years
130
100
0
0
100
200
73.5
64.6
55.7
46.9
38
29.1
20.3
11.4
2.52
-6.35
-15.2
-24.1
-33
-41.8
-50.7
-59.6
-68.4

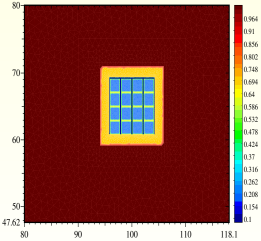
Pressure of gaz (bar) at T=1.000000e+04



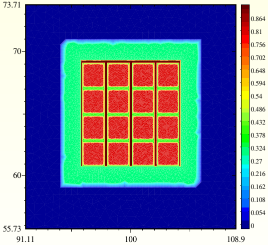
Pressure of water (bar) at T=1.000000e+04"years



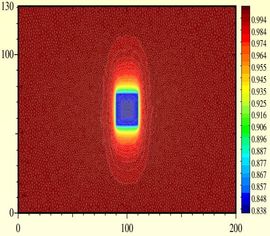
Saturation of water: step = 0, t=0.000000"years



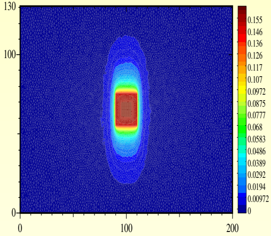
Saturation of gaz: step = 0, t=0.000000"years



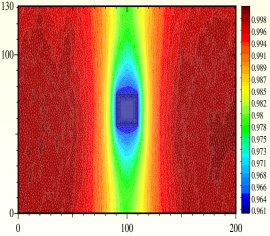
Saturation of water: t=510.000000"years



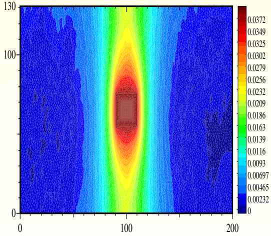
Saturation of gaz: t=510.000000"years



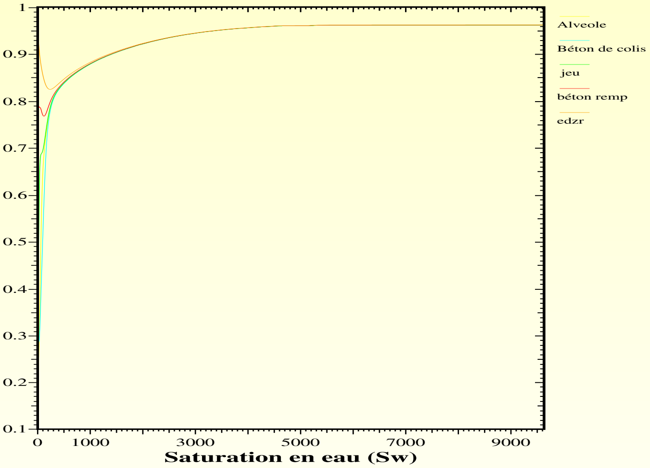
Saturation of water about 10 000 years



Saturation of gaz about 10000 years



Temporal evolution of water saturation



CONCLUSION ET PERSPECTIVES

- Développement d'un schéma IMPES EFMH-VF pour un écoulement diphasique sans échange.
- Application pour simuler le transfert du gaz autour d'un stockage des déchets nucléaires.
- Stabilité, estimations L^∞ et BV et Convergence.

Perspectives

- Choix d'un modèle compositionnel.
- Développement d'un schéma VF Implicite pour un écoulement diphasique eau-gaz avec échange
- Application : exercice Couplex-Gaz