Une présentation unifiée des méthodes de Galerkin Discontinu via les systèmes de Friedrichs

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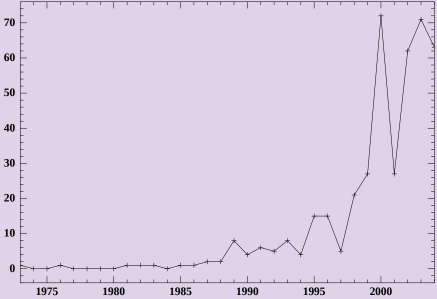
Outline

The development of DG methods

- Introduced in the 1970s
- Two somewhat parallel routes: hyperbolic or elliptic PDE's
- Time evolution of DG related papers



Hyperbolic PDE's

- Neutron transport simulation (Reed & Hill, '73)
- First abstract analysis (Lesaint & Raviart, '75)
  - based on Friedrichs' systems
  - analysis improvement (Johnson et al., '84)
- Recent developments ('00 onwards)
  - numerical fluxes, approximate Riemann solvers; Cockburn, Shu et al.
  - *hp*-adaptive DGFEM; Houston, Schwab, Süli et al.

Elliptic PDE's

- Interior Penalty (IP) to enforce continuity conditions
  - Nitsche, '71; Babuška & Zlámal, '73; Douglas & Dupont, '76; Baker, '77; Wheeler, '78; Arnold, '82
- Elliptic PDE's in mixed form
  - DG for primal variable only (Dawson, '93, '98)
  - DG for primal variable and flux (Bassi & Rebay, '97)
  - Local Discontinuous Galerkin (LDG) (Cockburn & Shu, '98)
- Non-symmetric variant of IP: NIPG (Baumann & Oden, '99; Oden et al., '98; Rivière et al., '99)

Towards a unified analysis of DG/IP methods

- Many methods share similar analysis tools
- First important step
  - Arnold, Brezzi, Cockburn, Marini, '00
  - Laplacian with homogeneous Dirichlet BC's
  - define numerical fluxes on mixed form
  - eliminate locally the flux

Goal of the present work

- Wider framework for unified analysis
- Encompass elliptic and hyperbolic PDE's

$\Longrightarrow$ Friedrichs' systems (FS) '58

- advection–reaction, advection–diffusion–reaction, Maxwell equations in diffusive regime, linear elasticity, wave equation, ...

Friedrichs' systems

- The setting
- The well–posedness theory
- Examples of FS

The setting

- FS are systems of first-order PDE's endowed with a symmetry and a positivity property
- The ingredients
  - Ω: bounded, open, connected, Lipschitz domain in $\mathbb{R}^d$
  - $m \geq 1$ (number of dependent variables)
  - $(d+1)$ $\mathbb{R}^{m,m}$-valued fields: $\mathcal{K}$ and $\{\mathcal{A}^k\}_{1\leq k\leq d}$
- Friedrichs' operator

$$T\psi = \mathcal{K}\psi + \underbrace{\sum_{k=1}^{d} \mathcal{A}^k \partial_k \psi}_{A\psi}$$

The four properties of FS

$$\mathcal{K} \in [L^\infty(\Omega)]^{m,m} \tag{A1}$$

$$\mathcal{A}^k \in [L^\infty(\Omega)]^{m,m} \quad \text{and} \quad \sum_{k=1}^{d} \partial_k \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m} \tag{A2}$$

$$\mathcal{A}^k = (\mathcal{A}^k)^t \ \text{ a.e. in } \Omega \tag{A3}$$

$$\exists \mu_0 > 0, \ \mathcal{K} + \mathcal{K}^t - \sum_{k=1}^{d} \partial_k \mathcal{A}^k \geq 2\mu_0 \mathcal{I}_m \tag{A4}$$

- Set $L = [L^2(\Omega)]^m$ and define the graph space

$$W = \{w \in L;\ Aw \in L\} \qquad \|w\|_W = \|Aw\|_L + \|w\|_L$$

- W is a Hilbert space and $T \in \mathcal{L}(W; L)$
- Formal adjoint $T^* \in \mathcal{L}(W; L)$

$$T^*\psi = \mathcal{K}^t\psi - \sum_{k=1}^{d} \partial_k(\mathcal{A}^k\psi)$$

Goal

Find a closed subspace $V \subset W$ such that $T : V \to L$ is an isomorphism

- This amounts to specifying BC's for the Friedrichs operator

The well–posedness theory

- Define $D \in \mathcal{L}(W; W')$ s.t.

$$\langle Du, v\rangle_{W',W} = (Tu, v)_L - (u, T^* v)_L$$

- Assume: $\exists M \in \mathcal{L}(W; W')$ s.t.

$$M \text{ is positive, i.e., } \langle Mw, w\rangle_{W',W} \geq 0,\ \forall w \in W \qquad \text{(M1)}$$

$$W = \mathrm{Ker}(D - M) + \mathrm{Ker}(D + M) \qquad \text{(M2)}$$

- Let $V = \mathrm{Ker}(D - M)$ and $V^* = \mathrm{Ker}(D + M^*)$

Main result

- Define

$$a(u,v) = (Tu,v)_L + \tfrac{1}{2}\langle (M-D)u,v\rangle_{W',W}$$

- The following system is well-posed

$$\begin{cases} \text{Seek } u \in W \text{ such that} \\ a(u,v) = (f,v)_L \quad \forall v \in W \end{cases}$$

- The unique solution satisfies $Tu = f$ and $u \in V$.

- Basis for designing the DG method

Examples of FS

- Advection–reaction
- Advection–diffusion–reaction
- Simplified 3D Maxwell's equations

Advection–reaction

$$\mu u + \beta \cdot \nabla u = f$$

- $\mu \in L^\infty(\Omega)$, $\beta \in [L^\infty(\Omega)]^d$, $\nabla \cdot \beta \in L^\infty(\Omega)$
- $\mu - \frac{1}{2}\nabla \cdot \beta \geq \mu_0 > 0$
- $m = 1$

$$\mathcal{K} = \mu \qquad \mathcal{A}^k = \beta^k$$

- The graph space is

$$W = \{w \in L^2(\Omega);\ \beta \cdot \nabla w \in L^2(\Omega)\}$$

Advection–reaction (cont'd)

- Let $\partial\Omega^\pm = \{x \in \partial\Omega;\ \pm\beta(x)\cdot n(x) < 0\}$
- Assume $\mathcal{C}^1(\overline{\Omega})$ dense in W and $\mathrm{dist}(\partial\Omega^-, \partial\Omega^+) > 0$
- Trace theorem

$$\langle Du, v\rangle_{W',W} = \int_{\partial\Omega} uv(\beta\cdot n)$$

- Suitable boundary operator M

$$\langle Mu, v\rangle_{W',W} = \int_{\partial\Omega} uv|\beta\cdot n|$$

yielding

$$V = \{v \in W;\ v|_{\partial\Omega^-} = 0\}$$
$$V^* = \{v \in W;\ v|_{\partial\Omega^+} = 0\}$$

Advection–diffusion–reaction

$-\Delta u + \beta\cdot\nabla u + \mu u = f$ in mixed form

$$\begin{cases} \sigma + \nabla u = 0 \\ \mu u + \nabla\cdot\sigma + \beta\cdot\nabla u = f \end{cases}$$

- Keep assumptions on μ and β
- $m = d + 1$

$$\mathcal{K} = \left[\begin{array}{c:c} \mathcal{I}_d & 0 \\ \hdashline 0 & \mu \end{array}\right] \qquad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & e^k \\ \hdashline (e^k)^t & \beta^k \end{array}\right]$$

- The graph space is $W = H(\mathrm{div};\Omega) \times H^1(\Omega)$

Advection–diffusion–reaction (cont'd)

- $\langle D(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle \tau\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega}(\beta\cdot n)uv$
- Suitable boundary operator M for Dirichlet BC's

$$\langle M(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tau\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}}$$

yielding $V = V^* = \{(\sigma,u)\in W;\ u|_{\partial\Omega} = 0\}$

- Neumann and Robin BC's can be treated as well

Simplified 3D Maxwell's equations

$$\begin{cases} \nu H + \nabla \times E = f \\ \sigma E - \nabla \times H = g \end{cases}$$

- $\nu, \sigma \in L^\infty(\Omega)$ uniformly bounded away from zero
- $m = 6$

$$\mathcal{K} = \left[\begin{array}{c:c} \nu \mathcal{I}_3 & 0 \\ \hdashline 0 & \sigma \mathcal{I}_3 \end{array}\right] \qquad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & \mathcal{R}^k \\ \hdashline (\mathcal{R}^k)^t & 0 \end{array}\right]$$

  $[\mathcal{R}^k \in \mathbb{R}^{3,3}]$
- The graph space is

$$W = H(\mathrm{curl};\Omega) \times H(\mathrm{curl};\Omega)$$

Simplified 3D Maxwell's equations (cont'd)

$$\langle D(H,E),(h,e)\rangle_{W',W} = (\nabla\times E,h)_{[L^2(\Omega)]^3} - (E,\nabla\times h)_{[L^2(\Omega)]^3} + (H,\nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H,e)_{[L^2(\Omega)]^3}$$

- Assume $[H^1(\Omega)]^3$ dense in $H(\mathrm{curl};\Omega)$
- Suitable boundary operator M to enforce $E\times n|_{\partial\Omega} = 0$

$$\langle M(H,E),(h,e)\rangle_{W',W} = -(\nabla\times E,h)_{[L^2(\Omega)]^3} + (E,\nabla\times h)_{[L^2(\Omega)]^3} + (H,\nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H,e)_{[L^2(\Omega)]^3}$$

yielding $V = V^* = H(\mathrm{curl};\Omega)\times H_0(\mathrm{curl};\Omega)$

Design and analysis of DG methods

- The discrete setting
- Design of the DG method
- Convergence analysis
- Applications

The discrete setting

- Shape-regular affine mesh family $\{\mathcal{T}_h\}_{h>0}$
- No matching assumption at interfaces
- Integer $p \geq 0$

$$W_h = \{v_h \in [L^2(\Omega)]^m;\ \forall K \in \mathcal{T}_h,\ v_h|_K \in [\mathbb{P}_p]^m\}$$
$$W(h) = [H^1(\Omega)]^m + W_h$$

- Set of interfaces $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\partial}$
  - jump $[\![\cdot]\!]$ and mean-value $\{\cdot\}$

- There exist two matrix-valued boundary fields s.t.

$$\langle Du, v\rangle_{W',W} = \int_{\partial\Omega} v^t \mathcal{D} u \qquad \text{with} \qquad \mathcal{D} = \sum_{k=1}^{d} n_k \mathcal{A}^k$$

$$\langle Mu, v\rangle_{W',W} = \int_{\partial\Omega} v^t \mathcal{M} u$$

provided u, v are smooth enough

- Extend matrix-valued field $\mathcal{D}$ to $\mathcal{F}_h$
  - $\mathcal{D}$ is two-valued on $\mathcal{F}_h^{\mathrm{i}}$

$$\mathcal{D} = \sum_{k=1}^{d} n_{K,k} \mathcal{A}^k \quad \text{a.e. on } \partial K$$

Design of the DG method

Two design ingredients

- boundary operators to enforce BC's weakly

$$\forall F \in \mathcal{F}_h^\partial, \qquad M_F \in \mathcal{L}([L^2(F)]^m, [L^2(F)]^m)$$

- interface operators to control jumps

$$\forall F \in \mathcal{F}_h^{\mathrm{i}}, \qquad S_F \in \mathcal{L}([L^2(F)]^m, [L^2(F)]^m)$$

Simpler setting based on matrix-valued fields $\mathcal{M}_F, \mathcal{S}_F \in \mathbb{R}^{m,m}$

$$M_F(v) = \mathcal{M}_F v \qquad S_F(v) = \mathcal{S}_F v$$

- General design conditions on M_F and S_F can be formulated
- Set of simpler conditions

Design of S_F

- S_F self-adjoint
- $S_F \sim 1$ ... more precisely, $\forall v \in [L^2(F)]^m$

$$c_1 \|\mathcal{D}v\|_{L,F}^2 \leq (S_F(v), v)_{L,F} \leq c_2 \|v\|_{L,F}^2$$

Design of M_F

- Consistency condition: $\forall v \in [L^2(F)]^m$,

$$(\mathcal{M}v - \mathcal{D}v = 0) \implies (M_F(v) - \mathcal{D}v = 0)$$

- $(M_F(v), v)_{L,F} \geq 0$; set $|v|^2_{M,F} = (M_F(v), v)_{L,F}$
- $|(M_F(v) - \mathcal{D}v, w)_{L,F}| \leq c|v|_{M,F}\|w\|_{L,F}$
- $|(M_F(v) + \mathcal{D}v, w)_{L,F}| \leq c\|v\|_{L,F}|w|_{M,F}$

The DG bilinear form

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h}(Tu,v)_{L,K} + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F(u) - \mathcal{D}u, v)_{L,F} - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}u\}, \{v\})_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}}(S_F[\![u]\!], [\![v]\!])_{L,F}$$

The discrete problem: For $f \in L$

$$\begin{cases} \text{Seek } u_h \in W_h \text{ s.t.} \\ a_h(u_h, v_h) = (f, v_h)_L \quad \forall v_h \in W_h \end{cases}$$

Local problems and the notion of flux

- $\forall K \in \mathcal{T}_h, \forall v_h \in \mathbb{P}_p(K)$

$$(u_h, T^* v_h)_{L,K} + (\phi_{\partial K}(u_h), v_h)_{L,\partial K} = (f, v_h)_{L,K}$$

- Element flux

$$\phi_{\partial K}(v)|_F = \begin{cases} \frac{1}{2} M_F(v|_F) + \frac{1}{2}\mathcal{D}v & F \subset \partial K^{\partial} \\ S_F([\![v]\!]_{\partial K}|_F) + \mathcal{D}_{\partial K}\{v\} & F \subset \partial K^{\mathrm{i}} \end{cases}$$

with cell-oriented jump

$$[\![z]\!]_{\partial K}(x) = \underbrace{z^i(x)}_{\text{interior}} - \underbrace{z^e(x)}_{\text{exterior}}$$

Convergence analysis

- Stability norm

$$\|v\|_{h,A} = \|v\|_L + |v|_J + |v|_M + \left(\sum_{K\in\mathcal{T}_h} h_K \|Av\|_{L,K}^2\right)^{\frac{1}{2}}$$

with

$$|v|_M^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (M_F(v), v)_{L,F} \qquad |v|_J^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F([\![v]\!]), [\![v]\!])_{L,F}$$

- Assume $\mathcal{A}^k \in [\mathcal{C}^{0,\frac{1}{2}}(\overline{\Omega})]^{m,m}$

- (Stability) $\exists c > 0$ s.t.

$$\inf_{v_h \in W_h \setminus \{0\}} \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{\|v_h\|_{h,A} \|w_h\|_{h,A}} \geq c$$

- (Continuity) $\exists c$ s.t.

$$\forall (v, w) \in W(h) \times W(h), \qquad a_h(v, w) \leq c \, \|v\|_{h,\frac{1}{2}} \|w\|_{h,A}$$

with

$$\|v\|_{h,\frac{1}{2}} = \|v\|_{h,A} + (\textstyle\sum_{K \in \mathcal{T}_h} [h_K^{-1} \|v\|_{L,K}^2 + \|v\|_{L,\partial K}^2])^{\frac{1}{2}}$$

Main result

Assume the exact solution u is in $[H^1(\Omega)]^m$. Then,

$$\|u - u_h\|_{h,A} \leq c \inf_{v_h \in W_h} \|u - v_h\|_{h,\frac{1}{2}}$$

▸ proof DG

- If $u \in W$ only, provided $[H^1(\Omega)]^m \cap V$ is dense in V

$$\lim_{h \to 0} \|u - u_h\|_L = 0$$

- Classical interpolation properties of DG space W_h
- If $u \in [H^{p+1}(\Omega)]^m$

$$\|u - u_h\|_{h,A} \leq c(u)h^{p+\frac{1}{2}}$$

- Convergence in L^2 of order $p + \frac{1}{2}$
- Optimal convergence in broken graph norm if mesh is quasi-uniform

$$(\sum_{K\in\mathcal{T}_h} \|A(u - u_h)\|^2_{L,K})^{\frac{1}{2}} \leq c(u)h^p$$

Applications

Advection–reaction

- $\mathcal{D}_{\partial K} = \beta{\cdot}n_K$, $\mathcal{M} = |\beta{\cdot}n|$
- Suitable choice: $\mathcal{M}_F = |\beta{\cdot}n|$ and for any $\alpha > 0$, $\mathcal{S}_F = \alpha|\beta{\cdot}n_F|$
- Element flux $\phi_{\partial K}(v)|_F$

$$\begin{cases} \frac{1}{2}\mathcal{M}v + \frac{1}{2}\mathcal{D}v = \frac{1}{2}|\beta{\cdot}n| + \frac{1}{2}(\beta{\cdot}n)v = (\beta{\cdot}n)^+ v \\ \mathcal{S}_F [\![v]\!]_{\partial K} + \mathcal{D}_{\partial K}\{v\} = \alpha|\beta{\cdot}n_F|(v^i - v^e) + \frac{1}{2}(\beta{\cdot}n_F)(v^i + v^e) \end{cases}$$

- Particular case ($\alpha = \frac{1}{2}$): recover well-known upwind flux

Advection–reaction–diffusion

- For Dirichlet BC's

$$\mathcal{D}_{\partial K} = \left[\begin{array}{c:c} 0 & n_K \\ \hdashline (n_K)^t & \beta\cdot n_K \end{array}\right] \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & -n \\ \hdashline n^t & 0 \end{array}\right]$$

- For $\alpha > 0$, $\eta > 0$

$$\mathcal{M}_F = \left[\begin{array}{c:c} 0 & -n \\ \hdashline n^t & \eta \end{array}\right] \qquad \mathcal{S}_F = \left[\begin{array}{c:c} \alpha n_F \otimes n_F & 0 \\ \hdashline 0 & \eta \end{array}\right]$$

- Penalizes jumps of $\sigma_h \cdot n$ and of u_h
- Neumann and Robin BC's can be treated as well

Simplified 3D Maxwell's equations

- Setting $E \times n|_{\partial\Omega} = 0$ yields

$$\mathcal{D}_{\partial K} = \left[\begin{array}{c|c} 0 & \mathcal{R}_K \\ \hline (\mathcal{R}_K)^t & 0 \end{array}\right] \qquad \mathcal{M} = \left[\begin{array}{c|c} 0 & -\mathcal{R}_K \\ \hline (\mathcal{R}_K)^t & 0 \end{array}\right]$$

with $\mathcal{R}_K \in \mathbb{R}^{3,3}$, $\mathcal{R}_K v = n_K \times v$

- For $\alpha > 0$, $\eta > 0$

$$\mathcal{M}_F = \left[\begin{array}{c|c} 0 & -\mathcal{R} \\ \hline \mathcal{R}^t & \eta\mathcal{R}^t\mathcal{R} \end{array}\right] \qquad \mathcal{S}_F = \left[\begin{array}{c|c} \alpha\mathcal{R}_F^t\mathcal{R}_F & 0 \\ \hline 0 & \eta\mathcal{R}_F^t\mathcal{R}_F \end{array}\right]$$

- Penalizes jumps of tangential components of H_h and E_h

Block FS and Local DG

- The setting
- Design of the LDG method
- Convergence analysis
- Applications

The setting

- Friedrichs' systems endowed with 2$\times$2 block structure
- Partition of dependent variable $z = (z^\sigma, z^u)$
  - z^σ can be eliminated
  - second-order (elliptic) PDE for z^u
- Examples
  - advection–diffusion–reaction $z^\sigma = \sigma$
  - simplified 3D Maxwell's equations $z^\sigma = H$ or $z^\sigma = E$

- $m = m_\sigma + m_u$, $L_\sigma = [L^2(\Omega)]^{m_\sigma}$, $L_u = [L^2(\Omega)]^{m_u}$

$$\mathcal{K} = \left[\begin{array}{c:c} \mathcal{K}^{\sigma\sigma} > 0 & \mathcal{K}^{\sigma u} \\ \hdashline \mathcal{K}^{u\sigma} & \mathcal{K}^{uu} \end{array}\right] \qquad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & \mathcal{B}^k \\ \hdashline (\mathcal{B}^k)^t & \mathcal{C}^k \end{array}\right]$$

- Set

$$B = \sum_{k=1}^d \mathcal{B}^k \partial_k$$
$$\tilde{B} = \sum_{k=1}^d [\mathcal{B}^k]^t \partial_k$$

- Elimination of z^σ

$$z^\sigma = [\mathcal{K}^{\sigma\sigma}]^{-1}\left(f^\sigma - \mathcal{K}^{\sigma u} z^u - B z^u\right)$$

- Second-order PDE for z^u

$$-\tilde{B}[\mathcal{K}^{\sigma\sigma}]^{-1} B z^u + \text{l.o.t.} = \text{r.h.s.}$$

- The above PDE is of elliptic type

Design of the LDG method

- Local DG method: eliminate discrete σ-component
- Polynomial degrees

$$p_u - 1 \leq p_\sigma \leq p_u \qquad 1 \leq p_u$$

- Approximation spaces

$$U_h = [P_{h,p_u}]^{m_u} \qquad \Sigma_h = [P_{h,p_\sigma}]^{m_\sigma} \qquad W_h = U_h \times \Sigma_h$$

- Local problems

$$\begin{cases} \text{Seek } z_h \in W_h \text{ s.t. } \forall q = (q^\sigma, q^u) \in [\mathbb{P}_{p_\sigma}(K)]^{m_\sigma} \times [\mathbb{P}_{p_u}(K)]^{m_u} \\ (z_h, T^* q)_{L,K} + (\phi_{\partial K}(z_h), q)_{L,\partial K} = (f, q)_{L,K} \end{cases}$$

Element fluxes

- $\phi_{\partial K}(z_h) = (\phi^\sigma_{\partial K}(z^u_h), \phi^u_{\partial K}(z^u_h, z^\sigma_h))$
- Elimination of z^σ_h by solving local problems
- $\Longrightarrow \phi^\sigma_{\partial K}$ only depends on z^u_h

- Boundary operator to weakly enforce BC's

$$M_F = \left[\begin{array}{c:c} 0 & M_F^{\sigma u} \\ \hdashline M_F^{u\sigma} & M_F^{uu} \end{array}\right] \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m)$$

- Interface operator to penalize jumps

$$S_F = \left[\begin{array}{c:c} 0 & S_F^{\sigma u} \\ \hdashline S_F^{u\sigma} & S_F^{uu} \end{array}\right] \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m)$$

- The jumps and boundary values of z^σ are no longer controlled

Convergence analysis

- General design conditions on S_F and M_F can be formulated
- Set of simpler conditions

Design of S_F

- $S_F^{\sigma\sigma} = 0$
- $S_F^{u\sigma} = 0$ and $S_F^{\sigma u} = 0$
- S_F^{uu} self-adjoint
- $S_F^{uu} \sim h_F^{-1}$ ... more precisely, $\forall v \in [L^2(F)]^{m_u}$

$$c_1(h_F \|\mathcal{D}^{uu} v\|^2_{L_u,F} + h_F^{-1} \|\mathcal{D}^{\sigma u} v\|^2_{L_\sigma,F}) \leq (S_F^{uu}(v), v)_{L_u,F}$$

$$(S_F^{uu}(v), v)_{L_u,F} \leq c_2 h_F^{-1} \|v\|^2_{L_u,F}$$

Design of M_F (Dirichlet BC's)

- Consistency conditions: $\forall y \in [L^2(F)]^m$

$$(\mathcal{M}y - \mathcal{D}y = 0) \implies (M_F(y) - \mathcal{D}y = 0)$$
$$(\mathcal{M}^t y + \mathcal{D}y = 0) \implies (M_F^*(y) + \mathcal{D}y = 0)$$

- $M_F^{\sigma\sigma} = 0$
- $M_F^{\sigma u}(v) = -\mathcal{D}^{\sigma u} v$ and $M_F^{u\sigma} = -(M_F^{\sigma u})^*$
- M_F^{uu} self-adjoint
- $M_F^{uu} \sim h_F^{-1}$ … more precisely, $\forall v \in [L^2(F)]^{m_u}$

$$c_1(h_F \|\mathcal{D}^{uu} v\|_{L_u,F}^2 + h_F^{-1} \|\mathcal{D}^{\sigma u} v\|_{L_\sigma,F}^2) \leq (M_F^{uu}(v), v)_{L_u,F}$$
$$(M_F^{uu}(v), v)_{L_u,F} \leq c_2 h_F^{-1} \|v\|_{L_u,F}^2$$

Design of M_F (Neumann or Robin BC's)

- Consistency conditions: $\forall y \in [L^2(F)]^m$

$$(\mathcal{M}y - \mathcal{D}y = 0) \implies (M_F(y) - \mathcal{D}y = 0)$$
$$(\mathcal{M}^t y + \mathcal{D}y = 0) \implies (M_F^*(y) + \mathcal{D}y = 0)$$

- $M_F^{\sigma\sigma} = 0$
- $M_F^{\sigma u}(v) = \mathcal{D}^{\sigma u} v$ and $M_F^{u\sigma} = -(M_F^{\sigma u})^*$
- M_F^{uu} self-adjoint
- $M_F^{uu} \sim 1$ ... more precisely, $\forall v \in [L^2(F)]^{m_u}$

$$c_1 \|\mathcal{D}^{uu} v\|^2_{L_u,F} \leq (M_F^{uu}(v), v)_{L_u,F} \leq c_2 \|v\|^2_{L_u,F}$$

- Stability norm

$$\|z\|_{h,A} = \|z^\sigma\|_{L_\sigma} + \|z^u\|_{L_u} + |z^u|_J + |z^u|_M + \left(\sum_{K\in\mathcal{T}_h}\|Bz^u\|^2_{L_\sigma,K}\right)^{\frac{1}{2}}$$

with $|z^u|_J^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}}(S_F^{uu}([\![z^u]\!]), [\![z^u]\!])_{L_u,F}$

- Assume $\mathcal{B}^k \in [\mathcal{C}^{0,1}(\overline{\Omega})]^{m_\sigma,m_u}$

Main result

Assume the exact solution z is in $[H^1(\Omega)]^m$. Then,

$$\|z - z_h\|_{h,A} \leq c \inf_{y_h \in W_h} \|z - y_h\|_{h,1}$$

with

$$\|z\|_{h,1} = \|z\|_{h,A} + \Big(\sum_{K \in \mathcal{T}_h} [h_K^{-2} \|z^u\|^2_{L_u,K} + h_K^{-1} \|z^u\|^2_{L_u,\partial K} + h_K \|z^\sigma\|^2_{L_\sigma,\partial K}]\Big)^{\frac{1}{2}}$$

- If $z \in W$ only, provided $[H^1(\Omega)]^m \cap V$ is dense in V

$$\lim_{h \to 0} \Big[\|z - z_h\|_L + \Big(\sum_{K \in \mathcal{T}_h} \|B(z^u - z_h^u)\|^2_{L_\sigma,K}\Big)^{\frac{1}{2}}\Big] = 0$$

- If $z \in [H^{p_\sigma+1}(\Omega)]^{m_\sigma} \times [H^{p_u+1}(\Omega)]^{m_u}$,

$$\|z - z_h\|_{h,A} \le c(z) h^{p_u}$$

- $p_\sigma = p_u$: suboptimal for $\|z^\sigma - z_h^\sigma\|_{L_\sigma}$ and $\|z^u - z_h^u\|_{L_u}$
- $p_\sigma = p_u - 1$: optimal for $\|z^\sigma - z_h^\sigma\|_{L_\sigma}$ and suboptimal $\|z^u - z_h^u\|_{L_u}$
- Improve $\|z^u - z_h^u\|_{L_u}$ by duality argument

The duality argument

- Let $\psi \in V^*$ solve $T^*\psi = (0, z^u - z_h^u)$ in L
- Assume elliptic regularity

$$\|\psi^u\|_{[H^2(\Omega)]^{m_u}} + \|\psi^\sigma\|_{[H^1(\Omega)]^{m_\sigma}} \leq c\|z^u - z_h^u\|_{L_u}$$

Main result

$$\|z^u - z_h^u\|_{L_u} \leq ch \inf_{y_h \in W_h} \|z - y_h\|_{h,1^+}$$

with $\|y\|_{h,1^+} = \|y\|_{h,1} + (\sum_{K\in\mathcal{T}_h}[h_K^2\|y^\sigma\|^2_{[H^1(K)]^{m_\sigma}} + h_K\|y^\sigma\|^2_{L_\sigma,\partial K}])^{\frac{1}{2}}$

Applications

Advection–diffusion–reaction

- For Dirichlet BC's

$$\mathcal{D}_{\partial K} = \left[\begin{array}{c:c} 0 & n_K \\ \hdashline (n_K)^t & \beta\cdot n_K \end{array}\right] \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & -n \\ \hdashline n^t & 0 \end{array}\right]$$

- For $\eta > 0$

$$\mathcal{M}_F = \left[\begin{array}{c:c} 0 & -n \\ \hdashline n^t & \eta h_F^{-1} \end{array}\right] \qquad \mathcal{S}_F = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \eta h_F^{-1} \end{array}\right]$$

- Penalizes jumps of u_h by h_F^{-1}
- LDG method by Cockburn and Shu '98
- Neumann and Robin BC's can be treated as well

The Laplacian

- Comparison with the unified analysis of Arnold, Brezzi, Cockburn and Marini, '02
- Lifting $r_F : [L^2(F)]^d \longrightarrow \Sigma_h$ s.t. $\|r_F(\tau_h)\|_{L_\sigma} \sim h_F^{-\frac{1}{2}} \|\tau_h\|_{L_\sigma,F}$
- IP (Douglas & Dupont, '76) [ζ and κ large enough]

$$M_F^{uu}(v) = \frac{\zeta}{h_F} v - r_F(vn_F)\cdot n_F \qquad S_F^{uu}(v) = \frac{\kappa}{h_F} v - \{r_F(vn_F)\} \cdot n_F$$

- Brezzi et al., '99 [ζ and κ positive]

$$M_F^{uu}(v) = \zeta r_F(vn_F)\cdot n_F \qquad S_F^{uu}(v) = \kappa \{r_F(vn_F)\} \cdot n_F$$

Simplified 3D Maxwell's equations

- Setting $E \times n|_{\partial\Omega} = 0$ yields

$$\mathcal{D}_{\partial K} = \left[\begin{array}{c|c} 0 & \mathcal{R}_K \\ \hline (\mathcal{R}_K)^t & 0 \end{array}\right] \qquad \mathcal{M} = \left[\begin{array}{c|c} 0 & -\mathcal{R}_K \\ \hline (\mathcal{R}_K)^t & 0 \end{array}\right]$$

- For $\eta > 0$

$$\mathcal{M}_F = \left[\begin{array}{c|c} 0 & -\mathcal{R} \\ \hline \mathcal{R}^t & \eta h_F^{-1} \mathcal{R}^t \mathcal{R} \end{array}\right] \qquad \mathcal{S}_F = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \eta h_F^{-1} \mathcal{R}_F^t \mathcal{R}_F \end{array}\right]$$

- Penalizes jumps of tangential components of E_h by h_F^{-1}

Conclusions

Friedrichs' systems

- The notion of symmetric systems goes beyond the traditional elliptic/hyperbolic classification of PDE's
- Boundary operators in FS are the natural way to enforce BC's in DG methods
- Extension of FS to the situation of partial coercivity
- Theory also applicable to linear elasticity, Stokes, and Oseen equations

DG methods

- Unified analysis for a large class of PDE's
- Design through operators M_F and S_F complying with a few general properties
- DG methods are stabilization techniques
- Natural link with cell-centered FV methods through the notion of fluxes

- Consistency. If $u \in [H^1(\Omega)]^m$,

$$\forall v_h \in W_h, \qquad a_h(u - u_h, v_h) = 0$$

- Second Strang Lemma

$$\begin{aligned} \|v_h - u_h\|_{h,A} &\le c \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(v_h - u_h, w_h)}{\|w_h\|_{h,A}} \\ &\le c \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(v_h - u, w_h)}{\|w_h\|_{h,A}} \le c\, \|u - v_h\|_{h,\frac{1}{2}} \end{aligned}$$