Maillages non conformes
et conditions d'interface arbitraires.
Application au cas à coefficients discontinus

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Arbitrary Interface Conditions and Non conforming grids

Model elliptic problem:

$$\eta p + \vec{a}.\nabla p - div(\kappa \nabla p) = f \quad \text{in } \Omega, \qquad p = g \quad \text{on } \partial\Omega.$$

Arbitrary transmission conditions as matching conditions can be useful in **optimized Schwarz (a.k.a two-field)** domain decomposition methods :

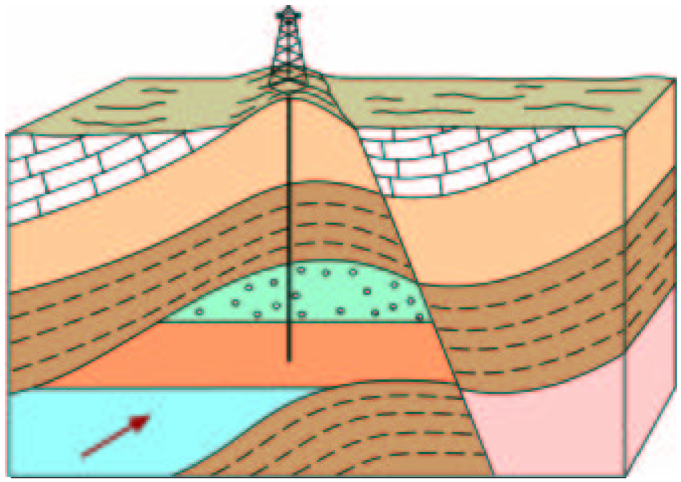
Iterative matching at the interface for a two subdomains decomposition:

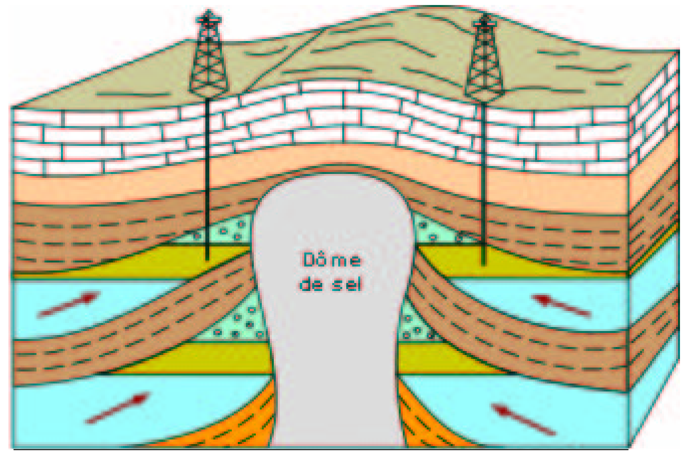
$$\kappa_1 \frac{\partial p_1^{n+1}}{\partial n_1} + S_1 p_1^{n+1} = -\kappa_2 \frac{\partial p_2^n}{\partial n_2} + S_1 p_2^n$$

$$\kappa_2 \frac{\partial p_2^{n+1}}{\partial n_2} + S_2 p_2^{n+1} = -\kappa_1 \frac{\partial p_1^n}{\partial n_1} + S_2 p_1^n$$

Problem : **How to discretize these conditions with finite volume and nonmatching grids?**

Example: Sedimentary Basin formation





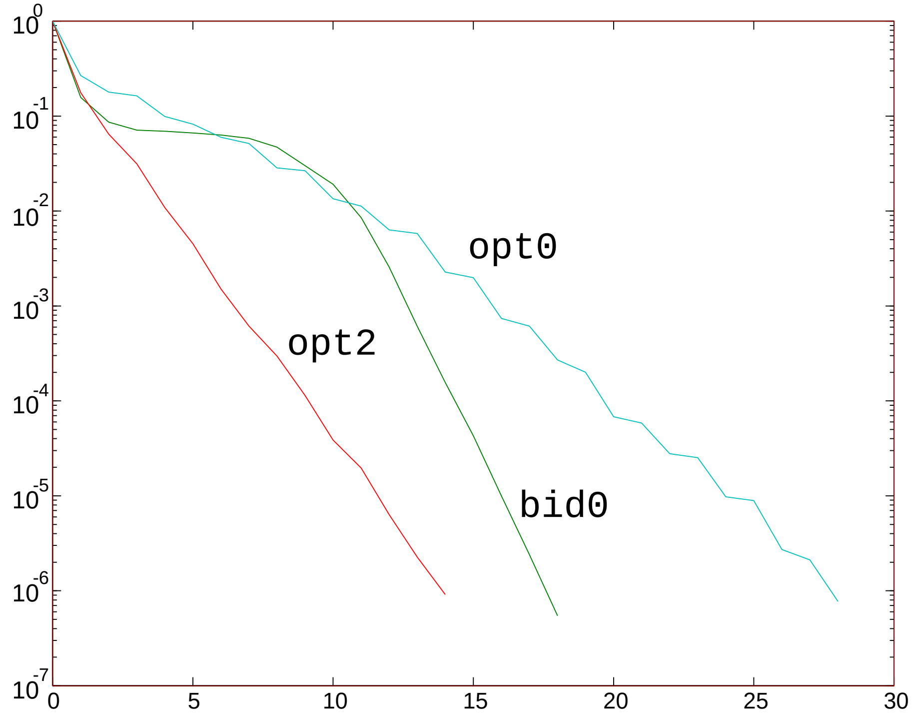
Location of oil, water, pressure in layers (drilling)

Conditions d'interface optimisées pour les MDD

travail en commun avec E. Flauraud

Construction algébrique de conditions d'interface (CI) optimisées

Sauts du coefficient κ de 4 ordres de grandeur.



résidu vs. nombre d'itérations

Comparaison entre des conditions de type Robin ($S = \alpha$) et des conditions d'ordre 2 ($S = \alpha - \beta\partial^2/\partial\tau^2$)

Valeurs propres du problème sous-structuré

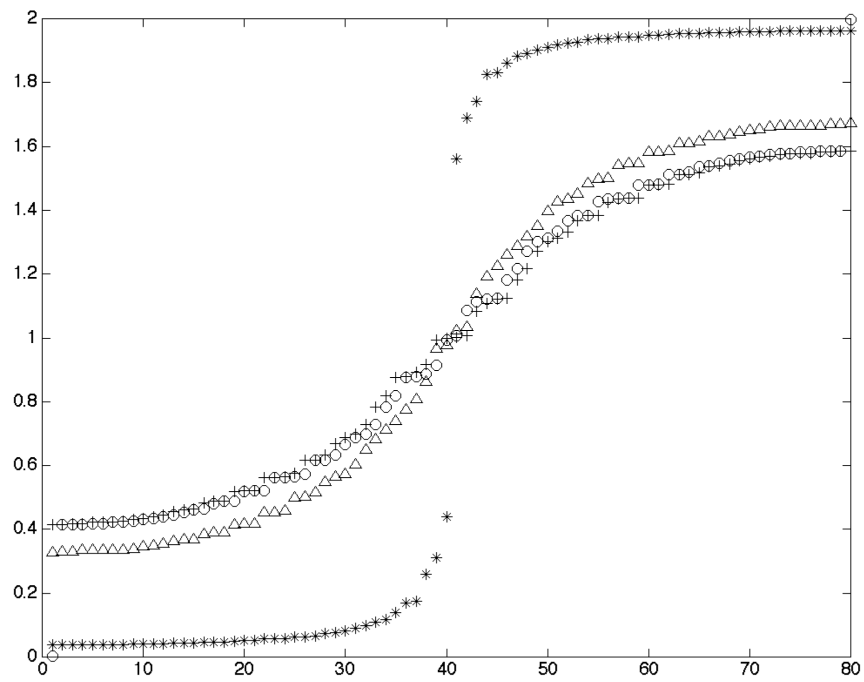
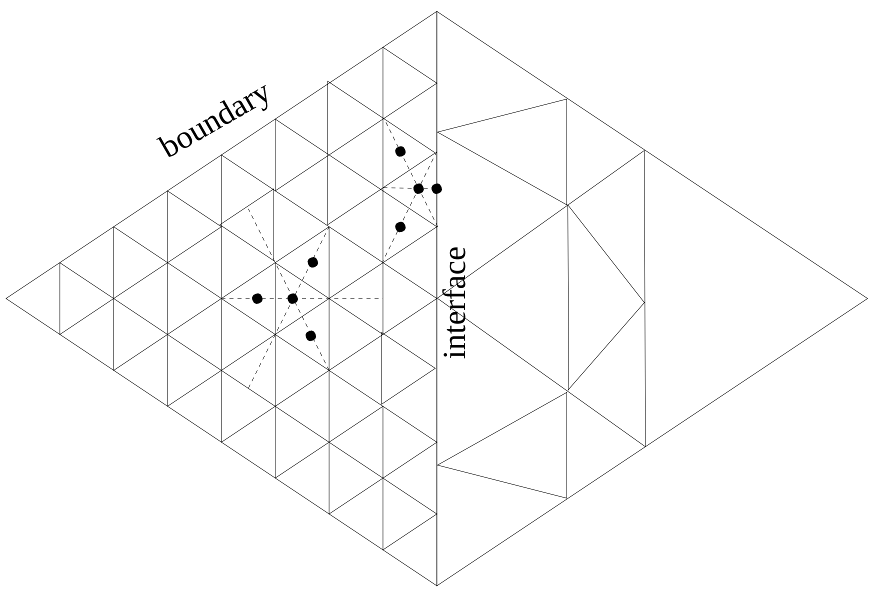


Figure 1: Conditions d'interface: star: opt0, triangle: opt2, circle: bid0, cross: bid2

Bibliography : Cell-Centered Finite Volume Schemes on the interfaces

Integrating the PDE in the volume K yields :

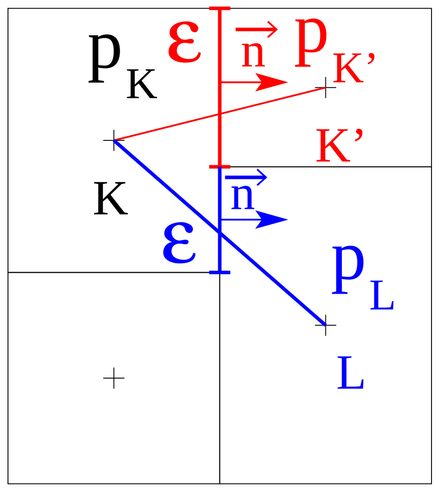
$$\int_K \eta p + \int_{\partial K} \vec{a}.n\, p - \int_{\partial K} \frac{\partial p}{\partial n} = \int_K f,$$



we shall call $(p_K)_{K \in \mathcal{T}}$ the approximation of $p(x_K)$ and $(p_\epsilon)_{\epsilon \in \mathcal{E}}$ the interface.

TPFA

$$u_{K,K'} \quad = \quad \frac{p_K - p_{K'}}{d(x_K, x_{K'})} meas(\epsilon)$$



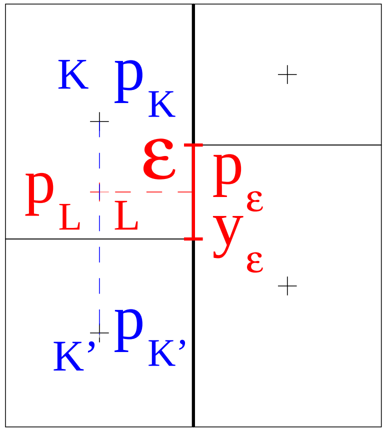
Consistency is lost because $[K', K]$ and $\vec{n}$ are not parallel. $F_{K,\epsilon}$ is a bad approximation of the outward flux.

- Scheme is stable.
- No consistency for outward flux through interface.
- Error Estimate: $O(h^{1/2})$ instead of $O(h)$ (classical FV). Cautrès-Herbin-Hubert.

Ceres (IFP)

$$p_L = \frac{meas([K,L])p_K + meas([K',L])p_{K'}}{meas([K,K'])}$$

$$u_\epsilon = \frac{p_\epsilon - p_L}{d(y_\epsilon, x_L)}$$



- Interpolation on subgrid inside the subdomains.
- No stability proven.
- Error Estimate seems to be in $O(h)$.

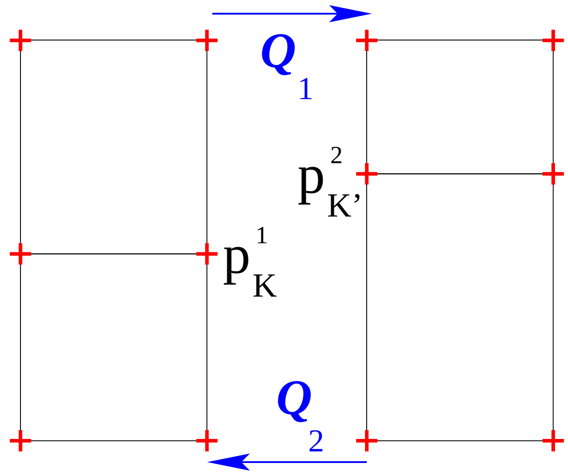
Mortar Method with Finite Element

Q_i are L^2 orthogonal projectors on the trace of FE of Ω_i and modify interface conditions in mortar conditions:

$$p_1 = Q_1(p_2)$$

$$u_2 = -Q_2(u_1)$$

They are no more symmetric:
Ω_1 is the slave, Ω_2 the master.

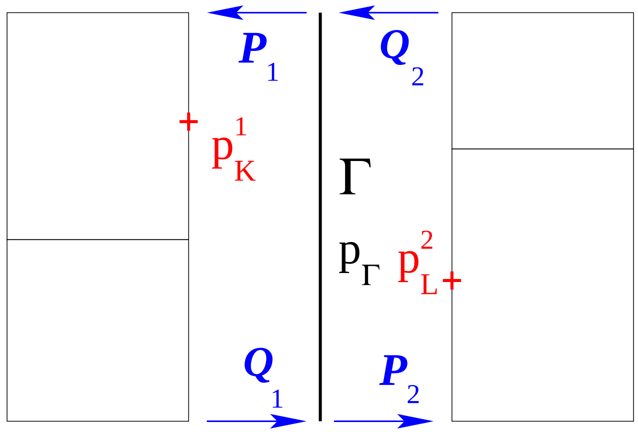


- Method based on finite element discretisation.
- Error estimate in $O(h)$ with P_1 FE.
- Dirichlet/Neumann Interface conditions type.

Bernardi-Maday-Patera.

Mortar Method with Mixed Finite Element

Mortar method extended to Mixed Finite Element (Mass conservation): a space of function is introduced in Γ.



Interface conditions are:

$$
\begin{aligned}
p_1 &= P_1(p_\Gamma) \\
p_2 &= P_2(p_\Gamma) \\
Q_1(u_1) &= Q_2(u_2)
\end{aligned}
$$

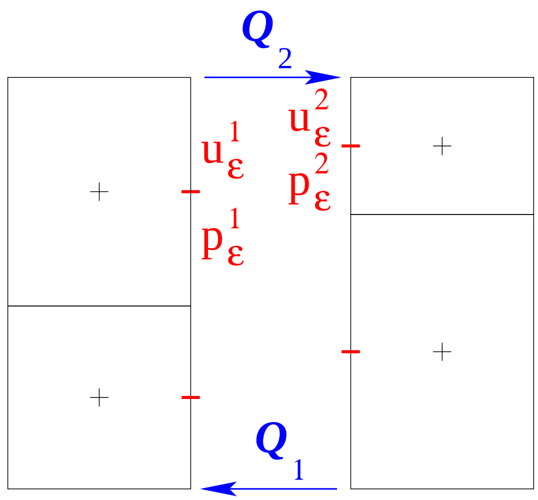
- Error estimate in $(O(h))$ with P1/P0 MFE.
- Dirichlet/Neumann interface conditions type.

Arbogast-Cowsar-Wheeler-Yotov.

New Cement

Mortar method extended to Robin interface conditions using FV (Q_i L^2 orthogonal projector).

The robin interface conditions are:



$$u_1 + \alpha_{glob} p_1 = Q_1(-u_2 + \alpha_{glob} p_2)$$

$$u_2 + \alpha_{glob} p_2 = Q_2(-u_1 + \alpha_{glob} p_1)$$

They are symmetric.

- Unique and global α: problem in heterogeneous media.
- Error estimate and solution depend on α ($\alpha_{opt} = O(1/h)^{1/2}$): $\alpha = O(1/h)^{\gamma} \Longrightarrow O(h^{1-\gamma/2})$

Achdou-Japhet-Maday-Nataf , *Numer. Math. 92 (2002)*

Goal of the new method

- Finite volume scheme.
- Non-Matching Grids.
- Error estimate in $O(h)$: as Finite Volume on Matching Grids.
- Arbitrary interface conditions.
- Generalization to heterogeneous media.

The finite volume method in the subdomains (R. Herbin, 95, Num. Meth. P.D.E)

- The domain Ω is partitioned into N non-overlapping subdomains.
- Let $\mathcal{T}_i$ be a partition of Ω_i made of polygonal closed sets K :

$$\bar{\Omega}_i = \cup_{K \in \mathcal{T}_i} K.$$

- $\mathcal{E}_{\Omega_i}$ is the set of the edges of $\mathcal{T}_i$.
- For two control volumes K and K' with $K \cap K' \in \mathcal{E}_{\Omega_i}$, let

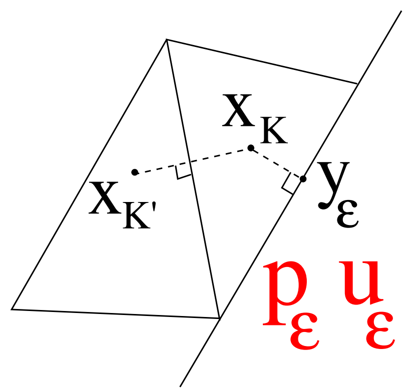
$$[K, K'] = \partial K \cap \partial K'.$$

- $\mathcal{E}_{iD}$ is the set of the edges on $\partial\Omega \cap \partial\Omega_i$: Dirichlet boundary conditions.
- $\mathcal{E}_i$ is the set of the edges on $\partial\Omega_i \backslash \partial\Omega$. **Transmission conditions** will be enforced on $\partial\Omega_i \backslash \partial\Omega$.

Assumption We suppose that there exist points $(y_\epsilon)_{\epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}}$ $(y_\epsilon \in \epsilon)$ and $(x_K)_{K \in \mathcal{T}_i} \in K$ such that

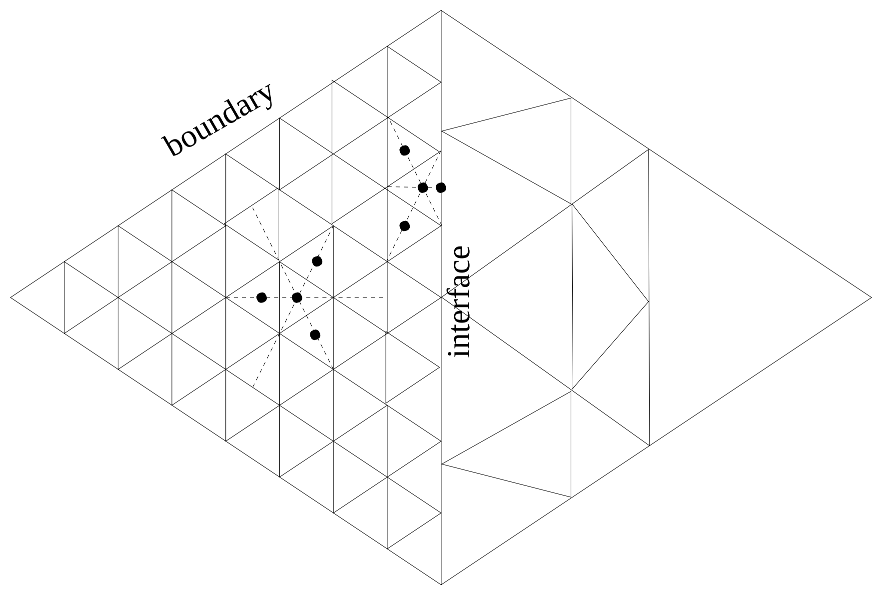
1. For two adjacent volumes K and K', the line $[x_K, x_{K'}]$ is orthogonal to the edge $[K, K']$.

2. For each edge $\epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}$, the straight line $[x_{K(\epsilon)}, y_\epsilon]$ is orthogonal to the edge ϵ.



The scheme in the subdomains Integrating the PDE in the volume K yields :

$$\int_K \eta p + \int_{\partial K} \vec{a}.n\, p - \int_{\partial K} \frac{\partial p}{\partial n} = \int_K f,$$



we shall call $(p_K)_{K\in\mathcal{T}}$ the approximation of $p(x_K)$ and $(p_\epsilon)_{\epsilon\in\mathcal{E}}$ the approximation of $p(y_\epsilon)$.

Discrétisation Volumes Finis

$$\int_K \eta p + \int_{\partial K} \vec{a}.n\, p - \int_{\partial K} \frac{\partial p}{\partial n} = \int_K f,$$

est discrétisé par

$$\eta meas(K) p_K^i - \sum_{K' \in \mathcal{N}_i(K)} \frac{p_{K'}^i - p_K^i}{d(x_{K'}, x_K)} meas([K, K'])$$

$$\sum_{K' \in \mathcal{N}(K)} a_{KK'} p_K^+ - \sum_{\epsilon \in \mathcal{E}_{iD}} \frac{g_\epsilon^i - p_K}{d(y_\epsilon, x_K)} meas(\epsilon) - \sum_{\epsilon \in \mathcal{E}_i} u_\epsilon^i meas(\epsilon) \quad = \quad F_K$$

avec p_K^+ décentrage amont, $a_{KK'} = \int_{[KK']} \vec{a} \dot{\vec{n}}_K$ et u_ϵ défini par

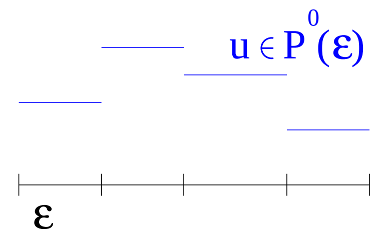
$$u_\epsilon^i \quad = \quad \frac{p_\epsilon^i - p_K^i}{d(y_\epsilon, x_K)} \text{ on } \epsilon \in \mathcal{E}_i$$

et avec des conditions d'interfaces sur $\mathcal{E}_i$ à définir liant $(u_\epsilon^i, p_\epsilon^i)$ avec $(u_\epsilon^j, p_\epsilon^j)$ où Ω_j est un sous-domaine voisin de Ω_i.

Dirichlet-Neumann interface conditions

Transmission operators Q_1 and Q_2 between the non conforming grids on the interface $\mathcal{E}_i \neq \mathcal{E}_j$:

$$Q_1 : P^0(\mathcal{E}_2) \longmapsto P^0(\mathcal{E}_1)$$
$$Q_2 : P^0(\mathcal{E}_1) \longmapsto P^0(\mathcal{E}_2)$$



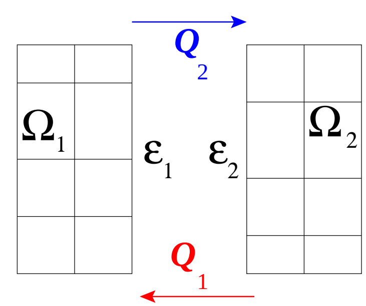
where $P^0(\mathcal{E}_i)$ space of piecewise constant functions $\mathcal{E}_i$.

Assumption 1 $\forall u \in P^0(\mathcal{E}_2)$ *and* $\forall v \in P^0(\mathcal{E}_1)$

$$< Q_1(u), v > \quad = \quad < u, Q_2(v) > \qquad (A1)$$

Dirichlet-Neumann Interface conditions (mortar type):

$$p_2 = Q_2(p_1)$$
$$u_1 = Q_1(-u_2)$$



Ω_1 is the master and Ω_2 the slave (conforming grids $Q_i = Id$).

Arbitrary Interface Conditions

Arbitrary Interface Conditions: $S_i : P^0(\mathcal{E}_i) \longmapsto P^0(\mathcal{E}_i)$

Assumption 2 *S_i is positive definite* $A2$

Arbitrary Interface Conditions:

$$\begin{aligned} Q_1(S_2(Q_2(p_1))) + u_1 &= Q_1(S_2(p_2) - u_2) \\ p_2 + Q_2(S_1^{-1}(Q_1(u_2))) &= Q_2(p_1 - S_1^{-1}(u_1)) \end{aligned}$$

Example of interface conditions:

- Steklov-Poincaré operator $(S_i = (DtN_i)_h)$
- Robin interface conditions $S_i = diag(\alpha_\epsilon^i)$, $S_i = diag(\alpha_{opt}^i)$ optimized of order 1 or 2 (S_i tridiagonal)

In New Cement, the interface relation was

$$u_1 + S_2(p_1) = Q_1(-u_2 + S_2(p_2))$$

Dirichlet/Neumann and arbitrary interface conditions

$$\left\{\begin{array}{lll} p_2 & = & Q_2(p_1) \quad (5) \\ u_1 & = & Q_1(-u_2) \quad (6) \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} S_2(p_2) & = & S_2(Q_2(p_1)) \\ S_1^{-1}(u_1) & = & S_1^{-1}(Q_1(-u_2)) \end{array}\right.$$

$$\Longrightarrow \left\{\begin{array}{rcll} Q_1(S_2(p_2)) & = & Q_1(S_2(Q_2(p_1))) & (5') \\ Q_2(S_1^{-1}(u_1)) & = & Q_2(S_1^{-1}(Q_1(-u_2))) & (6') \end{array}\right.$$

(6) + (5′) and (5) + (6′) yield arbitrary interface conditions

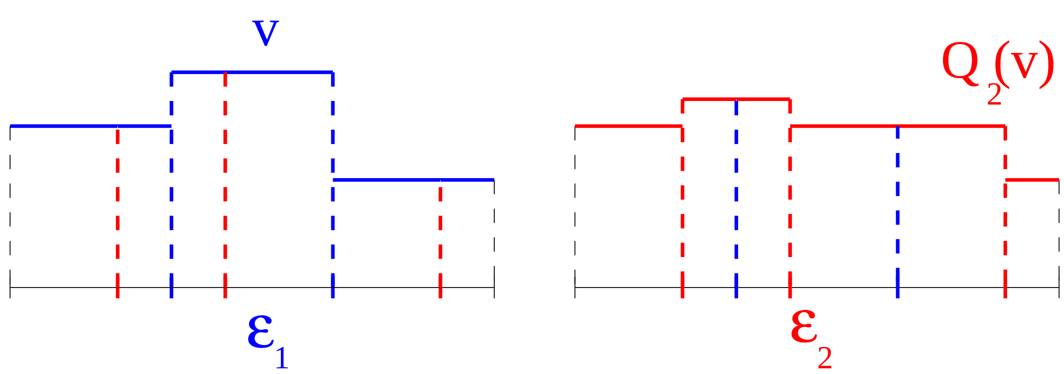
$$\begin{array}{rcll} Q_1(S_2(Q_2(p_1))) + u_1 & = & Q_1(S_2(p_2) - u_2) & (7) \\ p_2 + Q_2(S_1^{-1}(Q_1(u_2))) & = & Q_2(p_1 - S_1^{-1}(u_1)) & (8) \end{array}$$

$Q_i = P_i^C$ L^2 orthogonal projection on $P^0(\mathcal{E}_i)$

$\forall f \in L^2(\Gamma)$, $\forall \epsilon \in \mathcal{E}_i$, $Q_i(f) = \frac{1}{meas(\epsilon)} \int_\epsilon f$

$\forall u_j \in P^0(\mathcal{E}_j)$, $\forall \epsilon \in \mathcal{E}_i$, $i \neq j$,
$[Q_i(u_j)]_\epsilon = [P_i^C(u_j)] = \sum_{\epsilon' \in \mathcal{E}_j} \frac{meas(\epsilon \cap \epsilon')}{meas(\epsilon)} u^j_{\epsilon'}$

Operator Q_i satisfies assumption (A1).

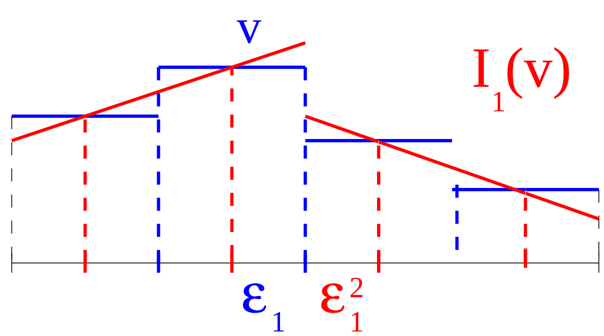


($Q_2(v)$ mean value on $\epsilon \in \mathcal{E}_2$ of $v \in P^0(\mathcal{E}_1)$)

Q_i defined via a linear rebuilding

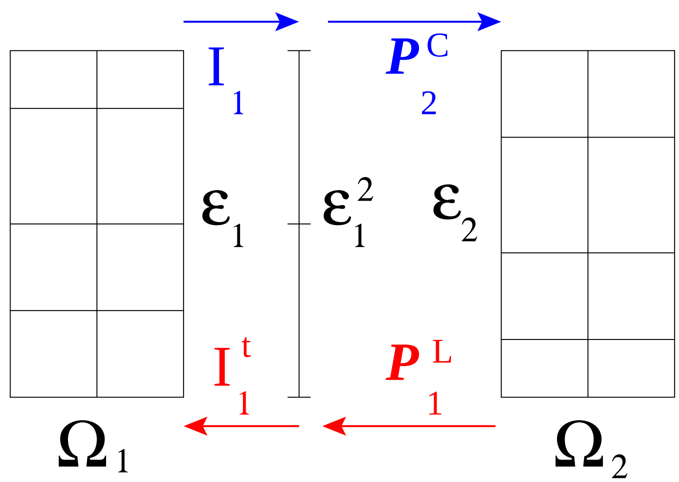
We introduce ($i = 1, 2$):

- the interface grid: $\mathcal{E}_i^2$ coarsening by a factor 2 of $\mathcal{E}_i$.
- $P_d^1(\mathcal{E}_i^2)$ discontinuous piecewise linear functions on $\mathcal{E}_i^2$.
- interpolation operator $I_i : P^0(\mathcal{E}_i) \longmapsto P_d^1(\mathcal{E}_i^2)$ and its tranpose I_i^t (w.r.t. the scalar product $L^2(\Gamma)$, $\forall u \in P^0(\epsilon_i)$ and $\forall v \in P^1(\epsilon_i^2)$ $< I_i(u), v >_{L^2(\Gamma)} =< u, I_i^t(v) >_{L^2(\Gamma)}$).



- P_i^L L^2 orthogonal projection on $P_d^1(\mathcal{E}_i^2)$

Transmission scheme



The definitions of the transmission operators are inspired by I. Yotov's work in mixed finite element method :

$$\begin{aligned} Q_2 &= P_2^C I_1 \\ Q_1 &= Q_2^t = I_1^t P_1^L \end{aligned}$$

They satisfy assumption (A1) (but are not projections).

Theoretical Results

- Globally and locally well-posed problems
- Dirichlet/Neumann and arbitrary interface conditions are equivalent
- Error estimate:
  - $O(h)^{1/2}$ with L^2 orthogonal projections
  - $O(h)$ with linear rebuilding
  - $O(h)$ with L^2 orthogonal projections if the master is a subgrid of the slave
  - Maximum principle with L^2 orthogonal projections if the slave is subgrid of the master.
- Error estimate done with a convective term as well

Global well posedness and stability estimate

Discrete Norm Definition:

$$
\begin{aligned}
||v||^2_{L^2(\Omega)} &= \sum_{K\in\tau}(v_K)^2 meas(K) \\
|v|^2_{1,\mathcal{T}} &= \sum_{K\in\tau}\sum_{K'\in\mathcal{N}(\mathcal{K})}\frac{(v_K - v_{K'})^2}{d(x_K, x_{K'})} meas([K, K']) \\
&+ \sum_{\epsilon\in\mathcal{E}_D}\frac{(v_K)^2}{d(x_K, y_\epsilon)} meas(\epsilon) + \sum_{\epsilon\in\mathcal{E}}\frac{(v_\epsilon - v_K)^2}{d(x_K, y_\epsilon)} meas(\epsilon)
\end{aligned}
$$

Global well posedness and stability estimate

Theorem 1 *Under assumptions A1 and A2, the global problem defined by the set of equations VF-(7)-(8) is well posed and there exists $C > 0$ independent of the mesh size such that:*

$$\sum_{i=1,2} [\eta||p_i||^2_{L^2(\Omega_i)} + |p_i|^2_{1,\mathcal{T}_i}] \quad \leq \quad C\sum_{i=1}^{2} ||F_i||^2_{L^2(\Omega_i)}$$

$\Longrightarrow$ The global discret problem has a unique solution without further assumption on the mesh

Local Well posedness

Theorem 2 *Under assumptions A1 and A2, the local problem defined in* Ω_1 *by the equations FV-(7) and the local problem defined in* Ω_2 *by the equations FV-(8) are well posed.*

$\Longrightarrow$ each local problem has an unique solution

For iteratively solving the domain decomposition problem, we have to compute local solution at each iteration.

Error Estimate

Discrete Norm Definition:

$$
\begin{aligned}
||v||^2_{L^2(\Omega)} &= \sum_{K \in \tau} (v_K)^2 meas(K) \\
|v|^2_{1,\mathcal{T}} &= \sum_{K \in \tau} \sum_{K' \in \mathcal{N}(\mathcal{K})} \frac{(v_K - v_{K'})^2}{d(x_K, x_{K'})} meas([K, K']) \\
&+ \sum_{\epsilon \in \mathcal{E}_D} \frac{(v_K)^2}{d(x_K, y_\epsilon)} meas(\epsilon) + \sum_{\epsilon \in \mathcal{E}} \frac{(v_\epsilon - v_K)^2}{d(x_K, y_\epsilon)} meas(\epsilon)
\end{aligned}
$$

General Error Estimate: For admissible meshes

If we define $e_K = p(x_K) - p_K$ and $e_\epsilon = p(y_\epsilon) - p_\epsilon$, with $p \in C^2(\Omega)$ solution of (2), $\exists C > 0$ such that:

$$
\left(\eta ||e||^2_{L^2(\Omega)} + |e|^2_{1,\mathcal{T}}\right)^{1/2} = Ch
$$

Error Estimate

Assumption 3 $\exists C, \beta > 0$ *such that:*

$$\forall \epsilon \in \mathcal{E}_i \ , \quad diam(\epsilon) \leq C d(x_K, y_\epsilon)^\beta$$

Theorem 3 *Under assumptions A1-A2-A3, we have:*

$$\left(\sum_{i=1,2} [\eta ||e_i||^2_{L^2(\Omega_i)} + |e_i|^2_{1,\mathcal{T}_i}] \right)^{1/2} \leq C(\Omega) h^\gamma \tag{1}$$

with $\gamma = 1/2$ *with* L^2 *orthogonal projections* ($\beta = 1$) *and* $\gamma = 1$ *with linear rebuilding* ($\beta = 1/2$).

Recall that

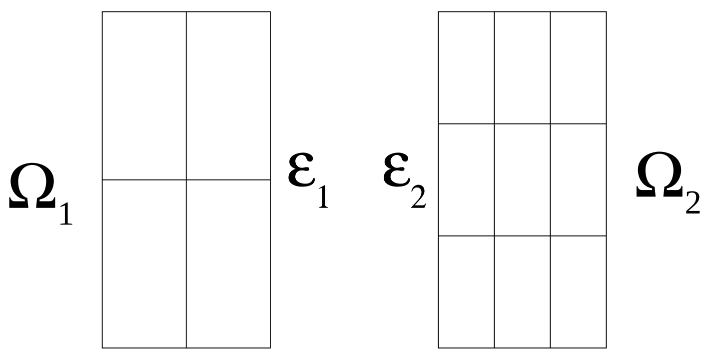
- $\gamma = 1$ with L^2 orthogonal projections if the master is a subgrid of the slave ($\beta = 1/2$).
- Maximum principle holds with L^2 orthogonal projections if the slave is a subgrid of the master.

Numerical Results in the homogeneous case

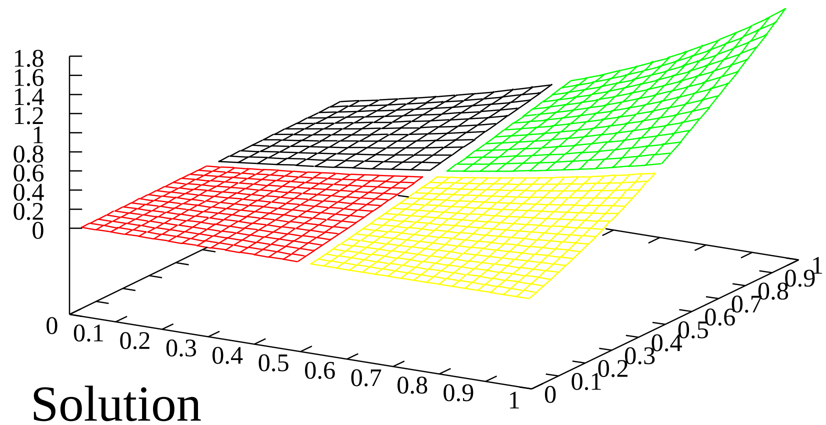
Numerical tests have been done with the equation in four subdomains :

$$\begin{aligned} p - \Delta p &= x^3y^2 - 6xy^2 - 2x^3 + (1 + x^2 + y^2)sin(xy) \text{ in } \Omega \\ p &= p_0 \text{ on } \partial\Omega \end{aligned}$$

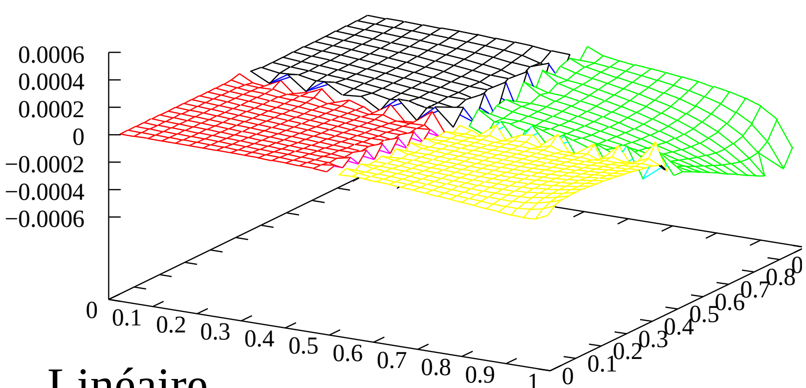
This results have been compared to the analytical solution which is $p(x, y) = x^3y^2 + sin(xy)$.
A substructuring method and a GMRES algorithm have been used.



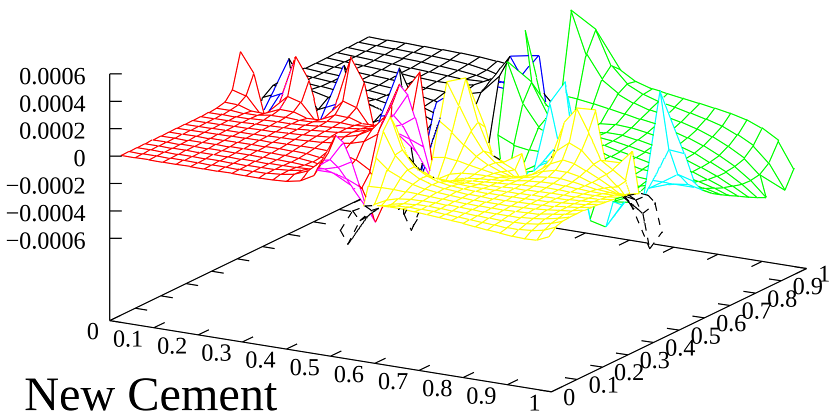
Solution and Error: 4 Domains



Solution

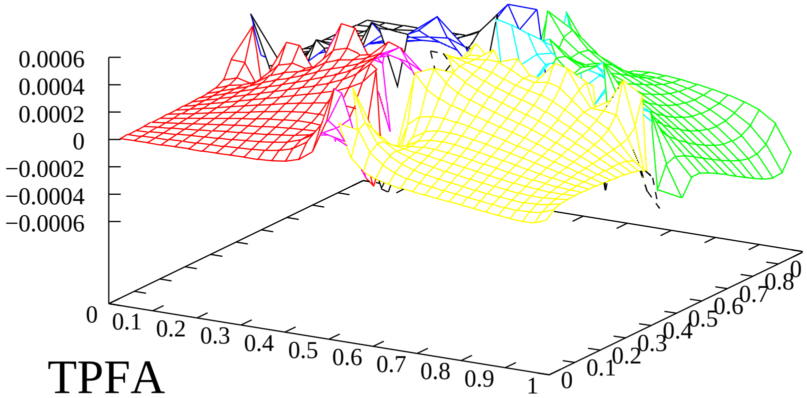


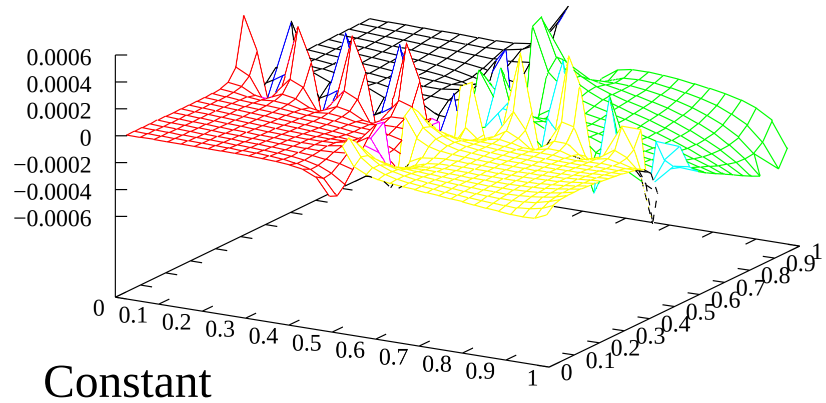
Linéaire



New Cement

Error: 4 Domains

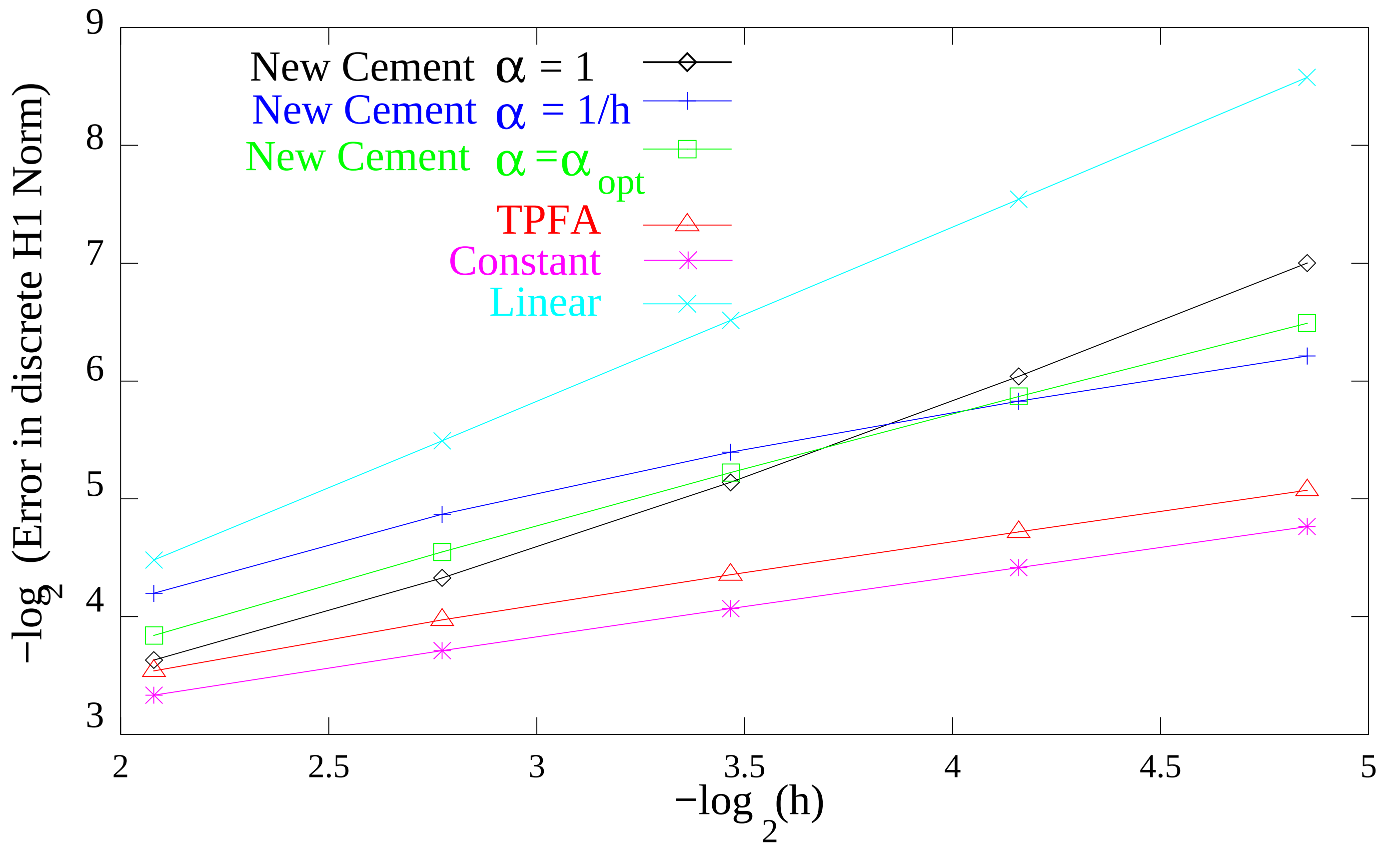




Numerical results obtained with $\alpha_{ij} = 1$ ($i \neq j$, et $i, j = \{1, 2, 3, 4\}$) and for grids

- Ω_1: 12×12 cells
- Ω_2: 14×14 cells
- Ω_3: 16×16 cells
- Ω_4: 18×18 cells

4 Domains

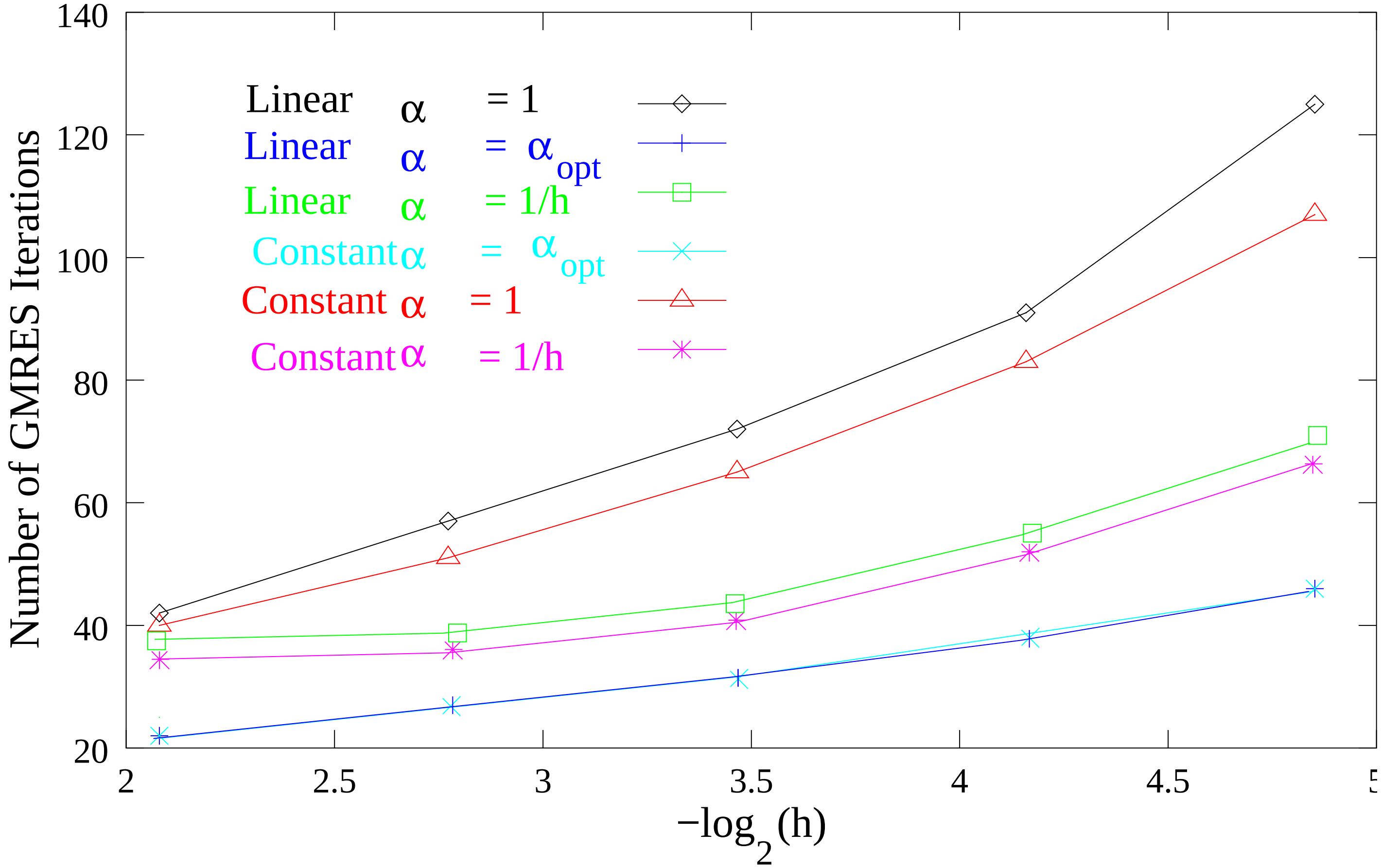


Linear $\simeq 1.3$; New Cement($\alpha = cte$)$\simeq 1.3$;

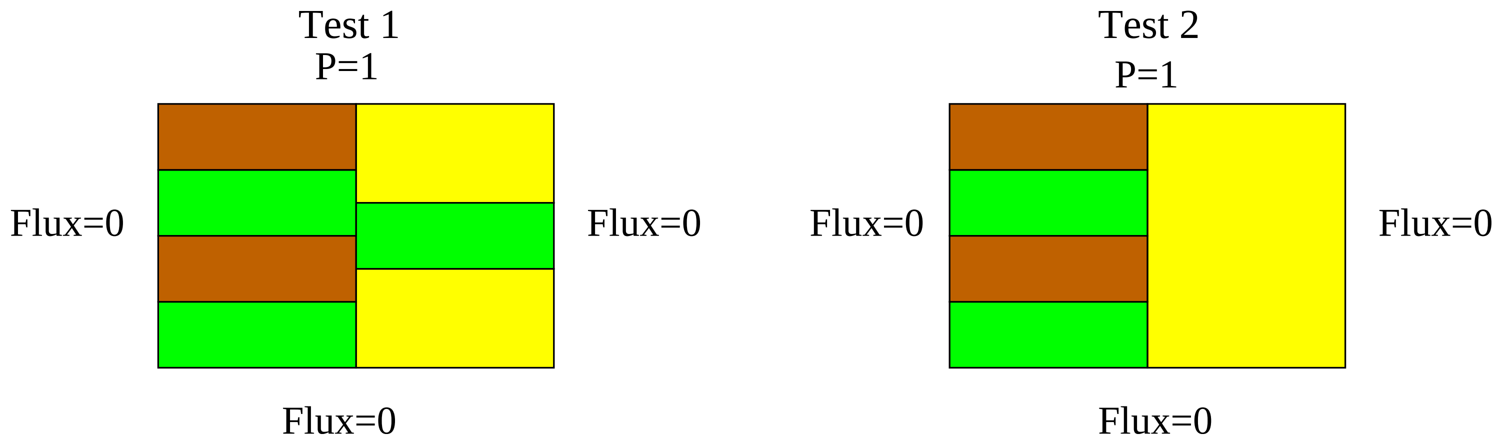
New Cement($\alpha = \alpha_{opt}$)$\simeq 0.9$;

New Cement($\alpha = 1/h$)$\simeq 0.6$ Constant$\simeq 0.5$; TPFA$\simeq 0.5$;

Iteration counts: 4 Domains



Heterogeneous Problem

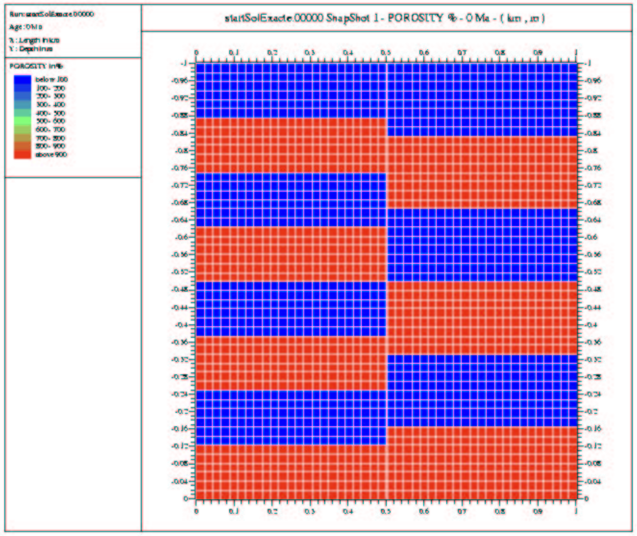


$$\eta p - div(\kappa \nabla p) = C$$

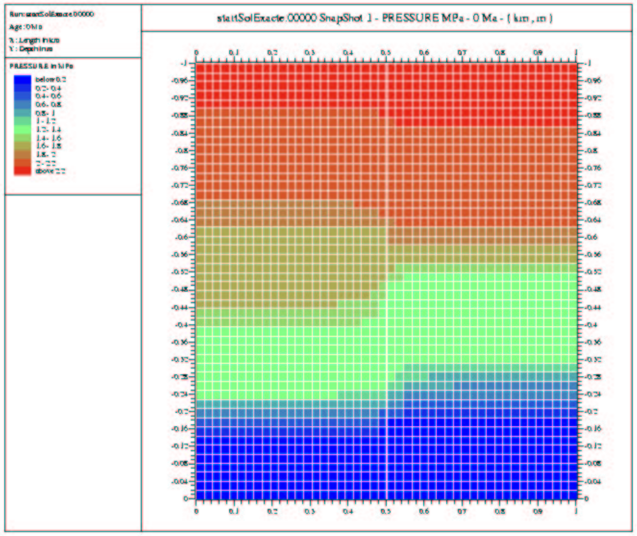
with η and κ highly discontinuous and anisotropic.

The flux across the interface is a highly discontinuous function. The points of discontinuities are located on the intersection points of the lithology. This feature has to be taken into account by the numerical scheme.

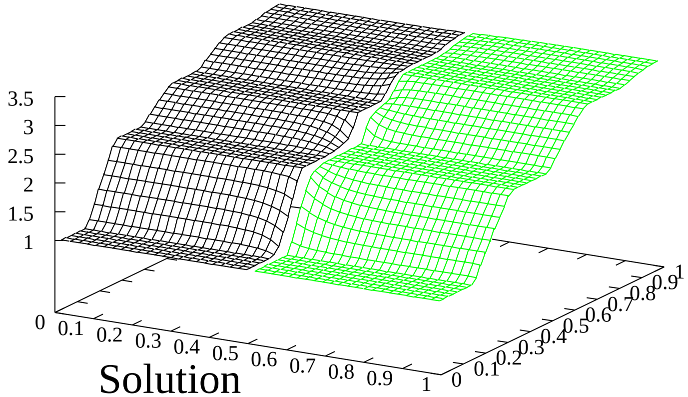
Non conforming heterogeneities



Hétérogénéité

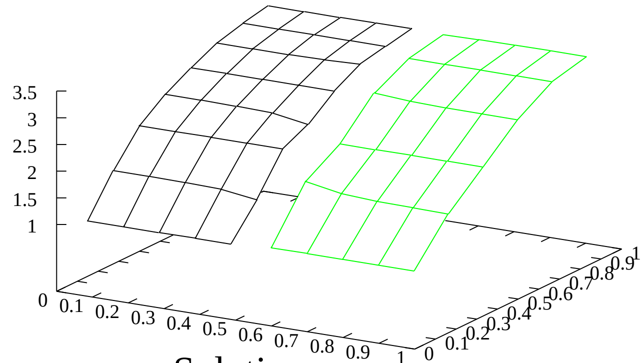


Solution

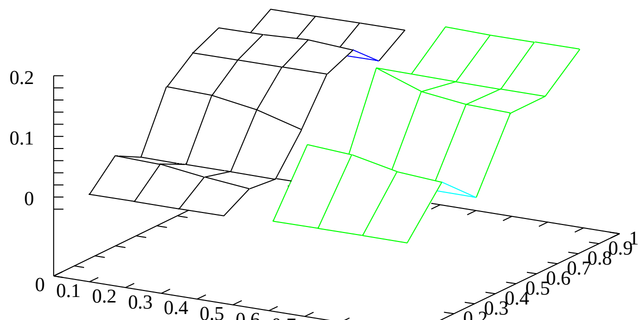


Solution

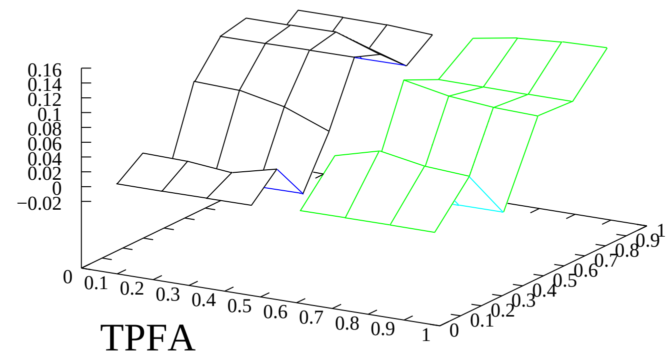
Basin Modeling



Solution

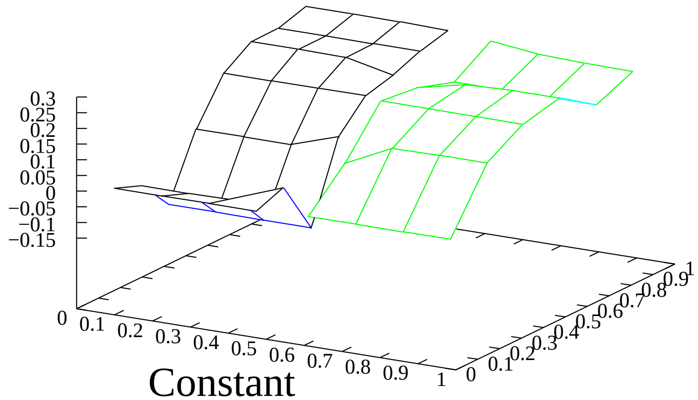


Ceres

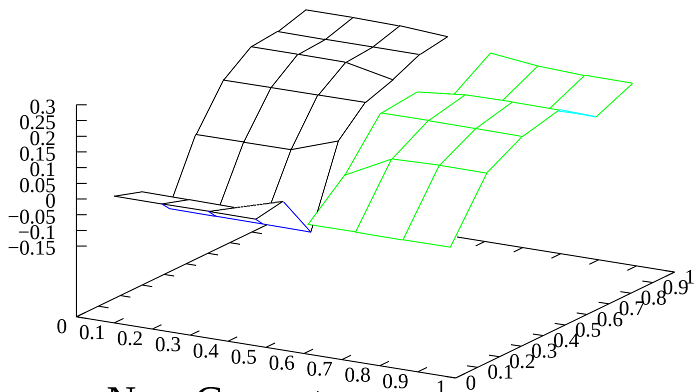


TPFA

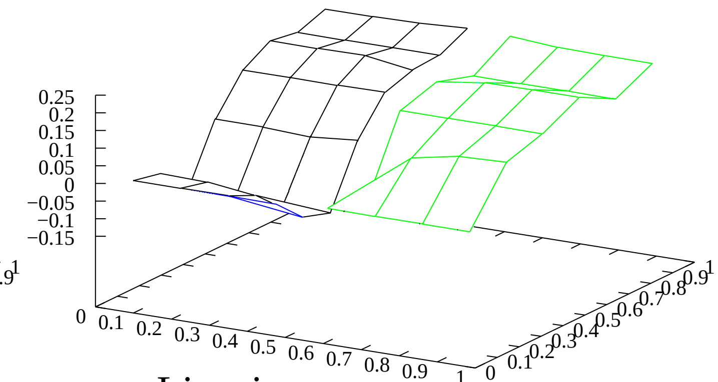
Basin Modeling



Constant



New Cement



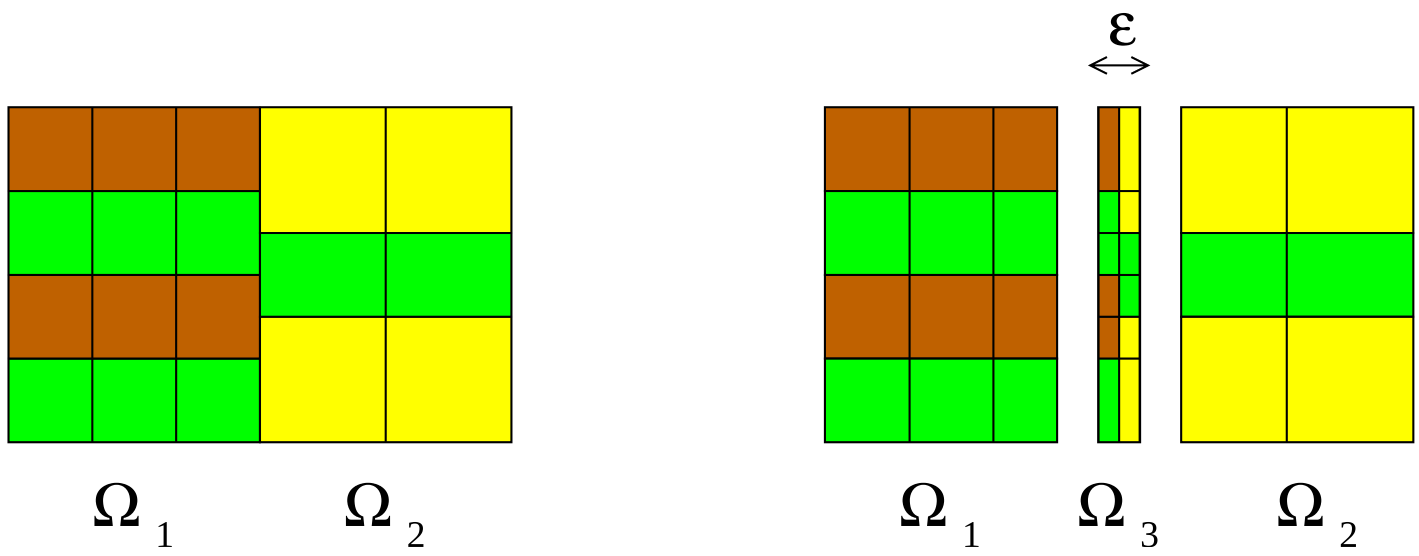
Lineaire

Tested Modifications

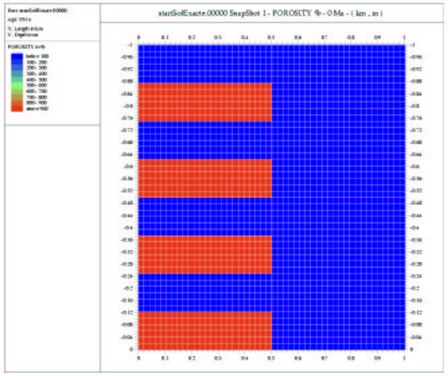
- Modified transmission operators that are κ dependant.

$$\Longrightarrow \text{KO}$$

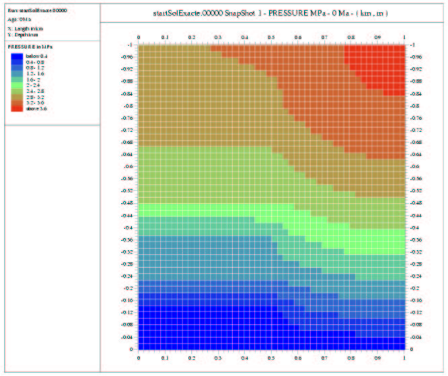
- A third subdomain is added in between ($\epsilon << 1$)



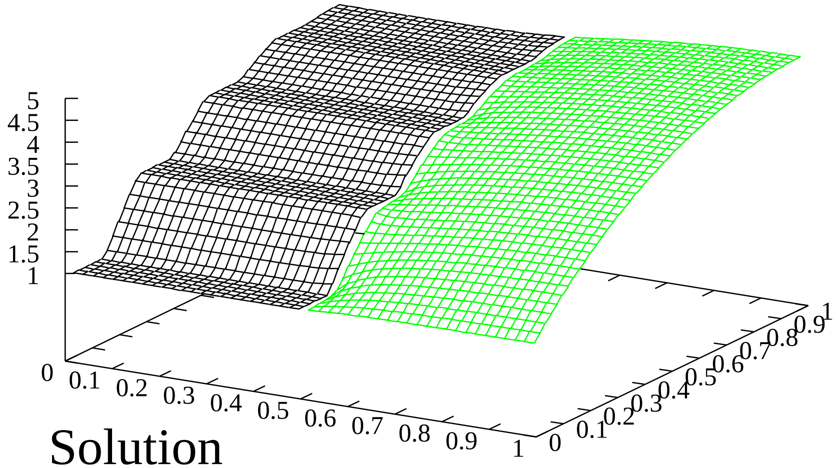
Results 2 domains/3 domains



Hétérogénéité



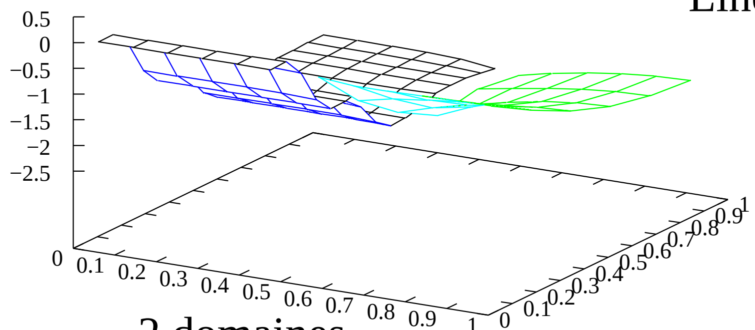
Solution



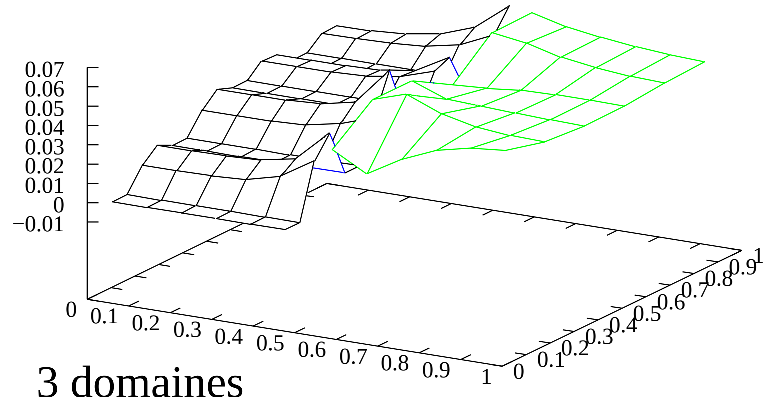
Solution

Linear and Ceres

Linéaire

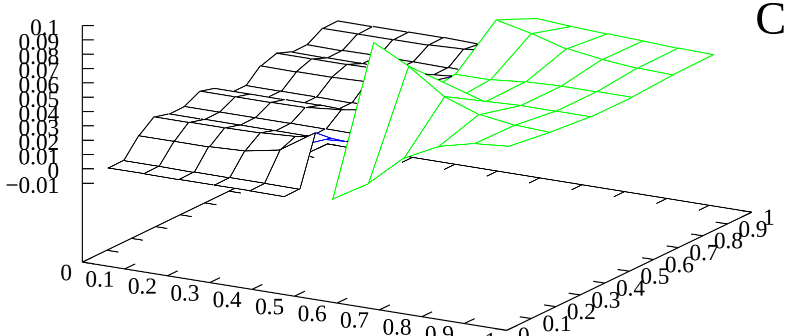


2 domaines

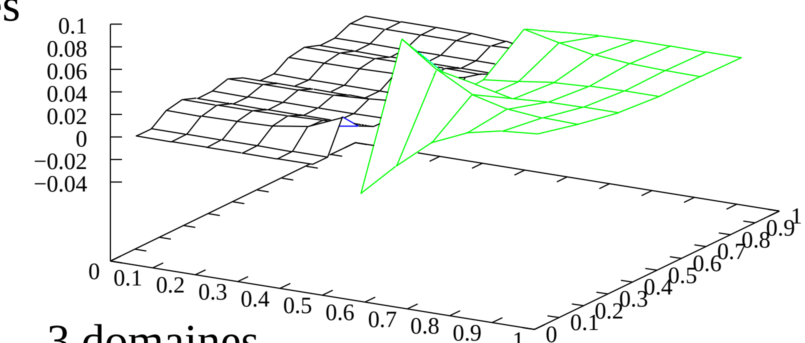


3 domaines

Ceres



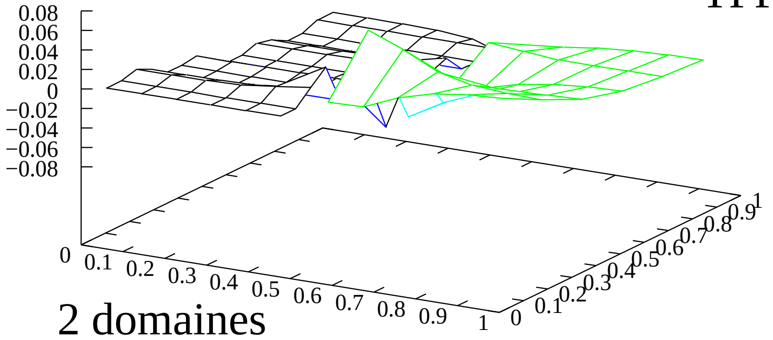
2 domaines

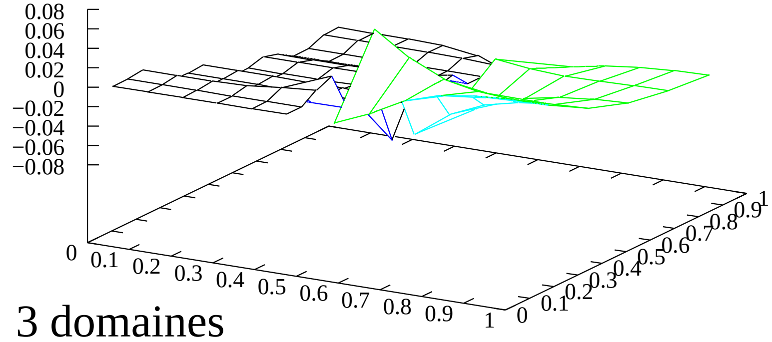


3 domaines

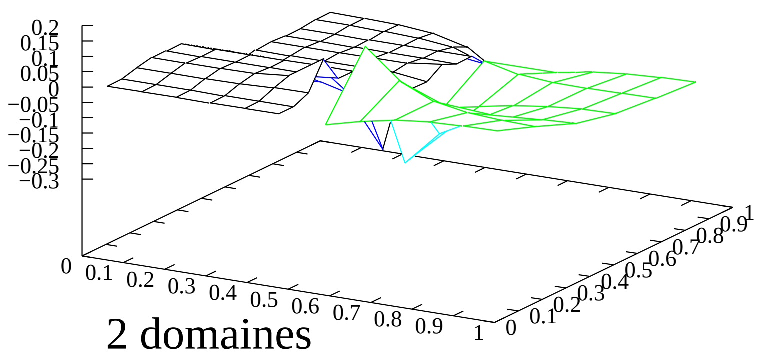
TPFA and Constant

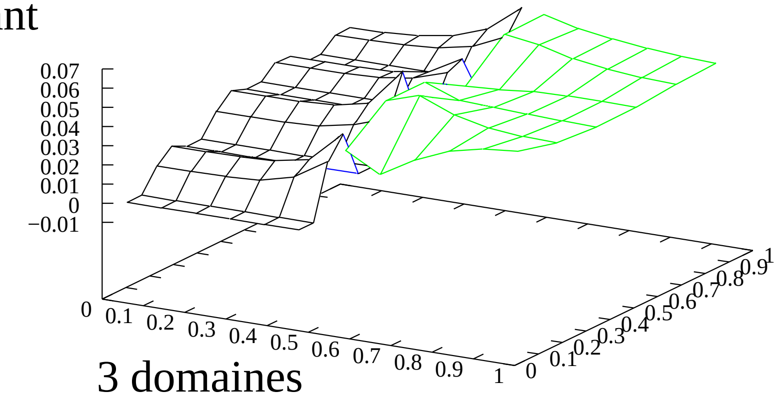
TPFA





Constant





Conclusion

- Method for arbitrary interface conditions
- Addition of a third subdomain for robustness w.r.t. heterogeneities

Prospects

- Analysis of the same idea for a Finite Element discretization
- Overlapping non conforming meshes

Thanks !