MODELING BY UPSCALING OF AN UNDERGROUND WASTE DISPOSAL SITE, POSSIBLY DAMAGED

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Experimental program in the Meuse/Haute Meuse URL

conceptual model of radionuclides transport

Tithonian: the only water resource

infiltration area

direct link with outlet (preliminary PA)

outlet area for diffusion

dominating process: diffusion

one-level disposal in the C-O mid-plan

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1 General model

\[ R\omega \frac{\partial \rho}{\partial t} - \nabla \cdot (A \nabla \rho) + (V \cdot \nabla) \rho + \lambda R\omega \rho = 0 \]  \hspace{1cm} (1)

- \( R \) the latency retardation factor,
- \( \omega \) the porosity,
- \( v \) the Darcy’s velocity
- \( \lambda = \frac{\log 2}{T} \); \( T \) the element radioactivity half life time
- according to the units width and their length we consider a storage 2D vertical section
- Iodine \(^{129}I\) has half life time \( T = 1.57 \times 10^7 \) years and is releasing during a time \( t'_m = 8 \times 10^3 \) years, with intensity \( \Phi' = 10^{-1} \).
With characteristic length $L \simeq 10^3 m$, and with characteristic time, the diffusion time $T_\alpha = \frac{\omega L^2 R}{|D|}$ in the host layer; With $x = \frac{x'}{L}$, $t = \frac{t'}{T_\alpha}$, then:

the thickness of the host layer and of the units are respectively of order $\varepsilon \simeq 10^{-1}$ and $\varepsilon^2 \simeq 10^{-2}$;

the rescaled releasing time, $t_m = \frac{t'_m}{T_\alpha} \simeq 10^{-3}$ is then very small compared to the total time $T = T/T_\alpha \simeq 1$ and to the half time of $^{129}I$, $\tau = \frac{T}{T_\alpha} \simeq O(1)$; but the leaking flux during this short releasing time is now, after renormalization, $\Phi = \Phi' \times L$, of order $10^2$. 
Figure 1: The three layers of soil containing the units
Figure 2: Units of Containers, after renormalization $x = \frac{x'}{L}$
2 The Equations

\[ \omega^e \frac{\partial \varphi^e}{\partial t} - \text{div} (A^e \nabla \varphi^e) + (v^e \cdot \nabla) \varphi^e + \lambda \omega^e \varphi^e = 0 \quad \text{in } \Omega^T_{e} (2) \]
\[ \varphi^e(0, x) = \varphi_0(x) \quad x \in \Omega^e \]
\[ n \cdot \sigma = n \cdot (A^e \nabla \varphi^e - v^e \varphi^e) = \Phi(t) \quad \text{on } \Gamma^T_{e} \]
\[ \varphi^e = 0 \quad \text{on } S^1, \]
\[ n \cdot (A^e \nabla \varphi^e - v^e \varphi^e) = 0 \quad \text{on } S^2 \]

with

\[ A^e(x_2) = A \left( \frac{x_2}{\varepsilon} \right); \quad v^e(x, t) = v(x, \frac{x_2}{\varepsilon}, t); \quad \omega^e(x_2) = \omega(x_2/\varepsilon). \]
(I) From the Disposal Units to a Global homogenized Repository model
3 A priori estimates

\[ \| \varphi_\varepsilon \|_{L^\infty(0,T;L^2(\Omega))} \leq C \quad (8) \]
\[ \| \varphi_\varepsilon \|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leq C \quad (9) \]

give

\[ \varphi_\varepsilon \rightharpoonup \varphi \quad \text{weak* in } L^\infty(0,T;L^2(\Omega)) \quad (10) \]
\[ \nabla \varphi_\varepsilon \rightharpoonup \nabla \varphi \quad \text{weakly in } L^2(0,T;L^{\beta*}(\Omega)) \quad \text{for } \beta < 2 \quad (11) \]
\[ \varphi_\varepsilon \rightharpoonup \varphi \quad \text{weak* in } L^2(0,T;M(\Omega)) \quad \text{for } \beta = 2 \quad (12) \]

\[ \beta^* = \frac{2\beta}{3\beta - 2}. \]
and $\varphi \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$;

$$\omega^2 \frac{\partial \varphi}{\partial t} - \text{div} (A^2 \nabla \varphi) + (v^2 \cdot \nabla) \varphi + \lambda \omega^2 \varphi = 0 \text{ in } \tilde{\Omega}^T$$  (13)

$$\varphi(x, 0) = \varphi_0(x) \quad x \in \tilde{\Omega} = \Omega \setminus \Sigma$$  (14)

$$\varphi = 0 \quad \text{on } S_1$$  (15)

$$n \cdot (A^2 \nabla \varphi - v^2 \varphi) = 0 \quad \text{on } S_2$$  (16)

$$[\varphi] = 0, \quad [e_2 \cdot (A^2 \nabla \varphi - v^2 \varphi)] = -|\tilde{M}| \Phi \quad \text{on } \Sigma,$$  (17)

where $[\cdot]$ denotes the jump over $\Sigma$, and $|\tilde{M}|$ stands for the limit of a normalized unit $M_\varepsilon$ area.
Figure 3: A renormalized unit $M_\epsilon$, in a period, after renormalization $y = x/\epsilon$; $S = 2|M| + O(\epsilon^{\beta-1}) = M_\epsilon$ area $\simeq |\widetilde{M}|$
Proof.

\[ 0 = - \int_{\Omega^T_\varepsilon} \omega^\varepsilon \varphi_\varepsilon \frac{\partial \psi}{\partial t} - \int_{\Omega_\varepsilon} \varphi_0 \psi(\cdot, 0) + \int_{\Omega^T_\varepsilon} A^\varepsilon \nabla \varphi_\varepsilon \nabla \psi + \]

\[ + \int_{\Omega^T_\varepsilon} (\mathbf{v}^\varepsilon \cdot \nabla) \varphi_\varepsilon \psi + \int_{\Omega^T_\varepsilon} \omega^\varepsilon \lambda \varphi_\varepsilon \psi - \int_0^T \Phi \sum_{i \in J(\varepsilon)} \int_{\Gamma^i_\varepsilon} \psi \, d\Gamma^i_\varepsilon; \]

\[ \int_{\Gamma^i_\varepsilon} \psi(x, t) \, d\Gamma^i_\varepsilon = (\psi(x^i_1, 0, t) + O(\varepsilon)) \mid \Gamma^i_\varepsilon \mid = \psi(x^i_1, 0, t) \mid \tilde{M} \mid \varepsilon + O(\varepsilon^\beta) , \]

\[ \sum_{i \in J(\varepsilon)} \int_{\Gamma^i_\varepsilon} \psi(x, t) \, d\Gamma^i_\varepsilon \rightarrow \mid \tilde{M} \mid \int_{\Sigma} \psi(x_1, 0, t) \, dx_1 . \]
Remark 1  We do not need periodicity in space, of the units. The same proof holds whenever each unit is randomly placed in a mesh of an $\varepsilon$–net. The units do not even need to have the same shape as long as their thickness is small enough ($\ll \varepsilon$).

We may extend to a general case where the flux $\Phi$ depends also on the space $\Phi(x,t)$ and the units have different shapes $M_\varepsilon(x)$, then the right hand side of (17) has to be replaced by $\lim_{\varepsilon \to 0} |M_\varepsilon(x)|\Phi(x',t)$.
4 Asymptotic expansion

Figure 4: $G_\varepsilon$ The inner layer; and $\Omega \setminus \overline{G_\varepsilon}$ the outer domain
In $G_{\varepsilon}$, the inner domain, we look for an asymptotic expansion of $\varphi_{\varepsilon}$:

$$
\varphi_{\varepsilon} \simeq \varphi_{0\varepsilon} + \varepsilon \left( \chi_{\varepsilon}^{k} \left( \frac{x}{\varepsilon} \right) \frac{\partial \varphi_{0\varepsilon}}{\partial x_{k}} + w_{\varepsilon} \left( \frac{x}{\varepsilon} \right) \Phi - \varphi_{0\varepsilon} \rho_{\varepsilon}^{k} \left( \frac{x}{\varepsilon} \right) v_{1k}^{1} \right) \equiv \varphi_{1\varepsilon} ,
$$

where $\varphi_{0\varepsilon}$ mimics the behaviour of $\varphi$ but has two jumps respectively on $\Sigma_{\varepsilon}^{+} = \{ \varepsilon \log (1/\varepsilon) \} \times ] - \delta/2, \delta/2 [$ and on $\Sigma_{\varepsilon}^{-} = \{-\varepsilon \log (1/\varepsilon) \} \times ] - \delta/2, \delta/2 [$, instead of only one on $\Sigma$. The functions $\chi_{\varepsilon}^{k}, \rho_{\varepsilon}^{k}$ and $w_{\varepsilon}$ are 1-periodic solutions in $y_{1}$ of three auxiliary stationary diffusion type problems posed in an infinite strip

$$
G_{\varepsilon} = ( ] - 1/2, 1/2[ \times \mathbb{R} ) \backslash \mathcal{M}_{\varepsilon} .
$$
The “micro shape corrector”:

\[-\text{div} \left( A \nabla (\chi^k_\varepsilon + y_k) \right) = 0 \text{ in } G_\varepsilon\]
\[ n \cdot A \nabla (\chi^k_\varepsilon + y_k) = 0 \text{ on } \partial M_\varepsilon ; \lim_{y_2 \to \infty} \nabla \chi^k_\varepsilon = 0, \quad (19)\]

Figure 5: The strip for local correctors; function \( \chi^2 \) along \( y_2 \) axis
The "source corrector":

\[- \text{div} (A \nabla \omega_\varepsilon) = 0 \text{ in } G_\varepsilon \]

\[ \mathbf{n} \cdot A \nabla \omega_\varepsilon = 1 \text{ on } \partial M_\varepsilon ; \lim_{y_2 \to \pm \infty} A \nabla \omega_\varepsilon (y) = \mp \frac{1}{2} |\partial M_\varepsilon| \mathbf{e}_2 \]

Figure 6: The source corrector 2D plot and plots along the $y_2$ axis for different $y_1$ positions
The “convection corrector”:

$$-	ext{div} \left( \mathbf{A} \nabla \rho^k_\varepsilon \right) = 0 \quad \text{in} \quad G_\varepsilon$$

$$\mathbf{n} \cdot \left( \mathbf{A} \nabla \rho^k_\varepsilon + \mathbf{e}_k \right) = 0 \quad \text{on} \quad \partial \mathcal{M}_\varepsilon; \quad \lim_{y_2 \to \infty} \nabla \rho^k_\varepsilon = 0. \quad (21)$$
4.1 Matched expansion and error estimate

With the approximation:

\[ F_\varepsilon = \begin{cases} 
\varphi_\varepsilon^0 & \text{in } \Omega \setminus \overline{G_\varepsilon} \text{(outer expansion)} \\
\varphi_\varepsilon^0 + \varepsilon \left( \chi_\varepsilon^k \frac{x_k}{\varepsilon} \frac{\partial \varphi_\varepsilon^0}{\partial x_k} + w_\varepsilon \left( \frac{x}{\varepsilon} \right) \Phi - \varphi_\varepsilon^0 \rho_\varepsilon^k \left( \frac{x}{\varepsilon} \right) v_1^k \right) & \text{in } G_\varepsilon.
\end{cases} \]  \hspace{1cm} (22)

**Theorem 1** For any \( 0 < \tau < 1 \) there exists a constant \( C_\tau > 0 \) non dependent on \( \varepsilon \), such that

\[ |\varphi_\varepsilon - F_\varepsilon|_{L^2(0,T;H^1(B_\varepsilon))} \leq C_\tau \varepsilon^\tau, \]  \hspace{1cm} (23)

where \( B_\varepsilon = \Omega \setminus \partial G_\varepsilon \).

The same estimate holds in \( L^\infty(0,T;L^2(\Omega_\varepsilon)) \) norm.
5 Conclusion

The expansion (22) clearly points out two important terms:

- the zero order term $\varphi_0$
- and the first order term $\varepsilon w_\varepsilon(\frac{x}{\varepsilon})\Phi$.

On one hand the diffusion in the low permeable layer around the units is small and on the other hand the containers are leaking intensively during a short time; then: during that short time the first order term $\varepsilon w_\varepsilon(\frac{x}{\varepsilon})\Phi$ will dominate in $\varphi_\varepsilon$; and after this short time the diffusion will become dominant, i.e. $\varphi_0$ is now the most important term in the expansion.

Remark: The effects of the boundary layer caused by the non periodicity of the geometry on $G_\varepsilon$ could be neglected for a $\varepsilon-$ order approximation.
Figure 7: 3D plot and 1D plot along the $y_2$ axis. Source term dominates. Discontinuity between inner and outer domain.
(II) Global homogenized Disposal Units model with a possibly damaged zone
Starting from a mathematical model describing the global behavior of one disposal unit of the underground waste repository,

Assuming it is made of a high number of containers sets, located inside a low permeable rock, lying on a hypersurface \( \Sigma \) and linked by parallel filled shafts, Fig:8; all the parallel shafts being connected at the top to a main shaft, also filled.

All the repository is embedded in a thin (100 m.) low permeability layer, called host layer, which is included between two higher permeability layers,

The convection field is given.
Figure 8: A part of a disposal unit, with 5 rows of containers sets and shafts
Denoting $\varepsilon$ the ratio between the width of a unit (500 m.) and distance (50 m.) between two shafts

- $\Rightarrow$ The containers set have a diameter, of order $\varepsilon^\gamma$, $\gamma$ close to three.

- $\Rightarrow$ In the renormalized model there are three scales: 1 for a disposal unit scale, $\varepsilon$ for both the scale of a containers row and the shafts period, and $\varepsilon^\gamma$ for the containers diameter.
Figure 9: Cell of periodicity $Y$ containing a shaft-damaged cylinder $S = ]-1/2,1/2[ \times C$ and a containers set $P_\varepsilon$; $\varepsilon^\gamma =$diameter of $P_\varepsilon$. 
6 The equations

\[ \mathbf{v}^\varepsilon(x) = \begin{cases} 
\mathbf{v}^h(x) & \text{in the host rock } \Omega_\varepsilon \setminus S_\varepsilon \\
\varepsilon^{-\beta} \mathbf{v}^d(x', x_2/\varepsilon; x_3/\varepsilon) & \text{in the shafts } S_\varepsilon 
\end{cases} \]

\[ \mathbf{A}^\varepsilon(x) = \begin{cases} 
\mathbf{A}^h(x) & \text{in the host rock } \Omega_\varepsilon \setminus S_\varepsilon \\
d(x) \mathbf{I} + \varepsilon^{-\beta} \mathbf{A}^d(x_2, x_2/\varepsilon, x_3/\varepsilon) & \text{in the shafts } S_\varepsilon 
\end{cases} \]

The convection in shafts goes only in the direction of the shafts \( \Rightarrow \)

\[ \mathbf{A}^d(x_2, y_2, y_3) = a(x_2, y_2, y_3) \left( \mathbf{e}_1 \otimes \mathbf{e}_1 \right) \]
"Microscopic" model of a disposal unit

\[ \omega^\varepsilon \frac{\partial \varphi^\varepsilon}{\partial t} - \text{div} (A^\varepsilon \nabla \varphi^\varepsilon) + (v^\varepsilon \cdot \nabla) \varphi^\varepsilon + \lambda \omega^\varepsilon \varphi^\varepsilon = 0 \quad \text{in} \ \Omega^\varepsilon_T \]  

(24)

\[ \varphi^\varepsilon(0, x) = \varphi_0(x) \quad x \in \Omega^\varepsilon \]  

(25)

\[ n \cdot (A^\varepsilon \nabla \varphi^\varepsilon - v^\varepsilon \varphi^\varepsilon) = \Phi^\varepsilon(t) \quad \text{on} \ \Gamma^\varepsilon_T \]  

(26)

\[ n \cdot (A^\varepsilon \nabla \varphi^\varepsilon - v^\varepsilon \varphi^\varepsilon) = \kappa (\varphi^\varepsilon - g^\varepsilon) \quad \text{on} \ \mathcal{K}^\varepsilon_T \cup \mathcal{H}^\varepsilon_T \]  

(27)

\[ \varphi^\varepsilon = 0 \quad \text{on} \ \mathcal{Z}^\varepsilon_T . \]  

(28)

with \( \mathcal{K}^\varepsilon_T \) the shafts cylindrical surfaces, \( \mathcal{H}^\varepsilon_T \) the shafts tops, \( \mathcal{Z}^\varepsilon_T \) the Shafts Bottoms and \( \Gamma^\varepsilon \) the Containers sets boundary \( \times (O, T) \).
7 Results

- $\beta < 1$

We assume for the boundary flux:
Existance of a continuous function $\Phi(t)$;

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma-1} \Phi_\varepsilon(t) = \Phi(t) \quad \text{uniformly in } t.$$ (29)

and

$$g_\varepsilon = g = \begin{cases} g^h & \text{on the shafts cylindrical surfaces } \mathcal{K}_\varepsilon \\ g & \text{on the shafts tops } \mathcal{H}_\varepsilon \end{cases}.$$
The shafts do not make any contribution, i.e. the repository behaves as if they were not there. \( \varphi_\varepsilon \to \varphi \) weakly in \( L^2(0, T; W^{1,\gamma^*}(\Omega)) \) and weak* in \( L^\infty(0, T; L^2(\Omega)) \), where \( \varphi \) is the unique solution of a problem, of same type as the microscopic problem:

\[
\omega^h \frac{\partial \varphi}{\partial t} - \text{div} (A^h \nabla \varphi) + (v^h \cdot \nabla) \varphi + \lambda \omega^h \varphi = 0 \quad \text{in } \tilde{\Omega}^T;
\]

\[
\varphi(0, x) = \varphi_0(x), \quad x \in \tilde{\Omega} = \Omega \setminus \Sigma; \quad \varphi = 0 \quad \text{on } S^T;
\]

\[
[\varphi] = 0, \quad \text{and} \quad [e_3 \cdot A^h \nabla \varphi - (v^h \cdot e_3) \varphi] = -\Phi \mathcal{M} \quad \text{on } \Sigma.
\]

where \( \tilde{\Omega}^T = (\Omega \setminus \Sigma) \times ]0, T[; S^T = \partial \Omega \times ]0, T[ \)

\[
[w](x') = w(x', 0+) - w(x', 0-) \), denotes the jump over \( \Sigma \) and \( \mathcal{M} \)
denotes the limit of the rescaled containers area, i.e.

\[
\mathcal{M} = \lim_{\varepsilon \to 0} \varepsilon^{1-\gamma} |\partial P_\varepsilon|.
\]
• $\beta = 1$

We assume for the behavior of the source term,

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(t) = \Phi(t) \text{ uniformly in } t,$$

while for $g_\varepsilon$ we suppose

$$g_\varepsilon = \begin{cases} 
g^h & \text{on the shafts cylindrical surfaces } \mathcal{K}_\varepsilon \\
\varepsilon^{-1} g^d & \text{on the shafts tops } \mathcal{H}_\varepsilon \end{cases}.$$
• $\beta = 1$

The processes, in and out of the damaged shafts are of same order and there are interactions between them.

$\varphi_\varepsilon \rightarrow \varphi_0$ weakly in $L^2(0, T; W^{1, \gamma^*}(\Omega))$ and $\varphi_\varepsilon \rightarrow \varphi^0 = \varphi(x_1, x_2, 0)$, $d\mu_\varepsilon(x) 2 - scale$, where $\varphi$ is the unique solution of a coupled problem

$$\omega^h \frac{\partial \varphi}{\partial t} - \text{div} (A^h \nabla \varphi) + (v^h \cdot \nabla) \varphi + \lambda \omega^h \varphi = 0 \text{ in } \tilde{\Omega}^T; \quad (34)$$

$$\varphi(0, x) = \varphi_0(x) \text{ in } \tilde{\Omega}; \quad (35)$$

$$n \cdot (A^h \nabla \varphi - v^h \varphi) = \kappa(\varphi - g^h) \text{ on } S^T$$

• we should construct the test functions that satisfy the Dirichlet condition on $Z_\varepsilon^\varepsilon$, starting from functions from $V$. 
• We need, for the integral

\[ \mathcal{L}^\varepsilon \psi = \varepsilon^{1-\gamma} \int_{\Gamma_\varepsilon} \psi \]

\[ \mathcal{L}^\varepsilon \in [H^1(\Omega)]', \text{ defined from } \psi \in H^1(\Omega) \quad (36) \]

to prove

\[ \mathcal{L}^\varepsilon \rightarrow \mathcal{M} \delta_\Sigma \text{ strongly in } [H^1(\Omega)]' . \]