FDEs and FPDEs: stability results, diffusive representations and applications

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DFA'10 : *Dynamiques Fractionnaires et Applications*.

— Tuesday, June 1st, 2010 —

Outline

The Lokshin model is non standard

$$\text{Let} \quad (\partial_t^2 + 2\,\eta\,\partial_t^{\frac{3}{2}} + \eta^2\,\partial_t^1)w - \partial_x^2\,w = 0, \quad t > 0, \quad x \in]0,1[$$

with init. cond. $w(t=0) = 0$ and $\partial_t w(t=0) = 0$,
and controlled dynamic boundary conditions at $x = 0$;
the system is being observed at $x = 1$.

- the damping is modelled by a fractional derivative.

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- the damping is modelled by a fractional derivative.
- there is no simple energy property, unlike in the classical cases of *fluid* ($\partial_t^1 w$) or *structural* ($-\partial_t^1\,\partial_x^2 w$) dampings.
- the spatial modes are no more orthogonal.
- in the case of the Webster-Lokshin model, the coefficients are variable with space : $\eta \mapsto \eta(x)$.

Outline

Analytic solution of a fractional PDE

Let $(\partial_t^2 + 2\,\eta\,\partial_t^{\frac{3}{2}} + \eta^2\,\partial_t^1)w - \partial_x^2\,w \;=\; 0, \quad t > 0, \quad x \in]0,1[$

with init. cond. $w(t=0) = 0$ and $\partial_t w(t=0) = 0$, and dynamical boundary conditions of *absorbing* type $(a_0 b_0 > 0, a_1 b_1 > 0)$ and controlled at $x = 0$:

$$\begin{cases} [a_0\,(\partial_t + \eta\,\partial_t^{\frac{1}{2}}) + b_0\,\partial_{-x})]\,w(t, x=0) \;=\; a_0\,(\partial_t + \eta\,\partial_t^{\frac{1}{2}})\,u(t) \\ [a_1\,(\partial_t + \eta\,\partial_t^{\frac{1}{2}}) + b_1\,\partial_x)]\,w(t, x=1) \;=\; 0 \end{cases}$$

with output : $y(t) \;=\; w(t, x=1)$. Then $y = h \star u$, with

$$h(t) = \sum_{n=-\infty}^{+\infty} c_n^\eta \left\{ \mathcal{E}_{\frac{1}{2}}(\sigma_n^{\eta+}, t) - \mathcal{E}_{\frac{1}{2}}(\sigma_n^{\eta-}, t) \right\} \in L^1(\mathbb{R}^+) \cap \mathcal{C}^\infty(\mathbb{R}^+)$$

where the $\sigma_n^{\eta\pm}$ are the roots of $\sigma^2 + \eta\sigma = s_n^0 = -\alpha^0 + 2i\pi f_n^0$.
BIBO stability comes from $\arg(\sigma_n^{\eta\pm}) > \frac{\pi}{4}, \forall n \in \mathbb{Z}$... Why ?

Fractional Differential Equations

For $0 < \alpha < 1$, consider the imput u – output y relation :

$$\sum_{k=0}^{p} a_k \, D^{k\alpha} y(t) = \sum_{l=0}^{q} b_l \, D^{l\alpha} u(t),$$

It is a *causal* pseudo-differential system, the symbol of which is, by Laplace transf. in some right-half plane $\mathbb{C}_a^+ := \Re e(s) > a$:

$$\mathcal{H}(s) = \frac{Q(s^\alpha)}{P(s^\alpha)} \quad \text{avec} \quad \left| \begin{array}{lcl} Q(\sigma) & \triangleq & \displaystyle\sum_{l=0}^{l=q} b_l \, \sigma^l \\ P(\sigma) & \triangleq & \displaystyle\sum_{k=0}^{k=p} a_k \, \sigma^k \end{array} \right. .$$

Necessary and Sufficient Stability Condition

From the input-output viewpoint, the BIBO-stability result is as follows, $y = h \star u$, with :

Theorem

$$\textit{BIBO stability} \quad \Longleftrightarrow \quad \begin{cases} q \leq p \\ |\arg \sigma| > \alpha \frac{\pi}{2}, \quad \forall \sigma \in \mathbb{C}, \,/\, P(\sigma) = 0 \end{cases}$$

In which case, h has the long memory asymptotics :

$$h(t) \sim K\, t^{-1-\alpha} \quad \textit{as} \quad t \to +\infty.$$

Sketch of the proof

Algebraic tools can be used, since the orders are commensurate to the same α : let P and Q two coprime polynomials, and let $R = Q/P$ the rational function, we get the structure result :

Proposition

$$h(t) = \sum_{n=1}^{N} \sum_{m=1}^{m_n} r_{nm} \, \mathcal{E}_{\alpha}^{\star m}(\lambda_n, t),$$

with $R(\sigma) = \sum_{n=1}^{N} \sum_{m=1}^{m_n} r_{nm} \, (\sigma - \lambda_n)^{-m}$.

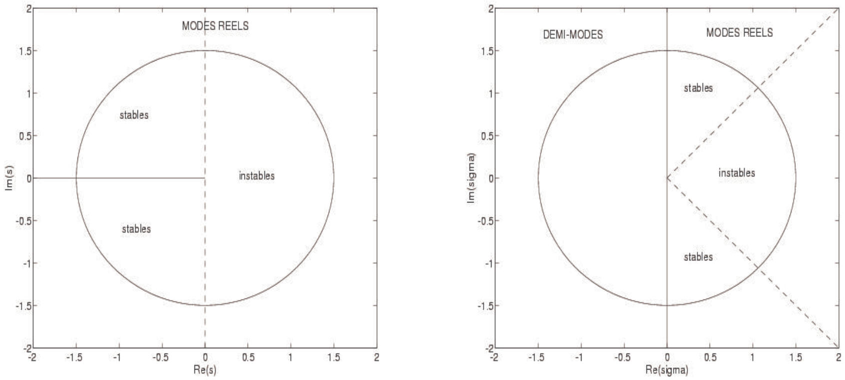
where $\mathcal{E}_{\alpha}^{\star m}(\lambda, t)$ is a Mittag-Leffler function (a hypergeometric special function), the LT of which is $(s^{\alpha} - \lambda)^{-m}$.

For $\alpha = 1$, it reduces to the well-known causal polynomial–exponential $\frac{1}{m!} \, t^{m-1} \, \exp(\lambda \, t)$.

N. & S. Stability condition : an illustration

Stability of $E_\alpha(\lambda\, t^\alpha)$ with LT $s^{\alpha-1}(s^\alpha - \lambda)^{-1}$, as a fct. of $\arg(\lambda)$.

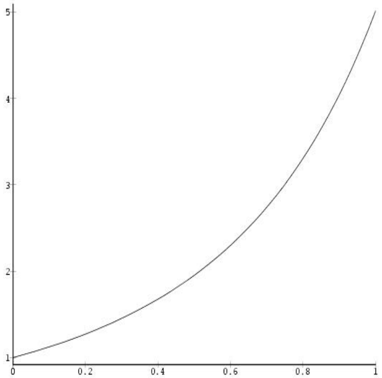
Laplace plane : s σ-plane



Mittag-Leffler functions in $\mathbb{C}$ (I)

$$t \mapsto E_\alpha(\lambda\, t^\alpha) \text{ for } \alpha = \frac{1}{2} \text{ and } \arg(\lambda) = 0$$

Real part

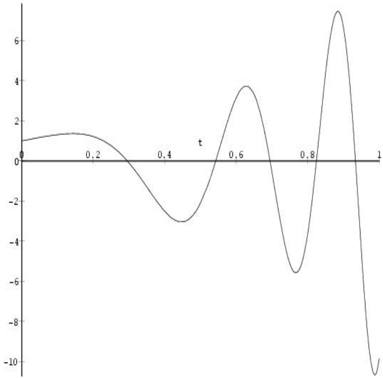


Imaginary part

Mittag-Leffler functions in $\mathbb{C}$ (II)

$$t \mapsto E_\alpha(\lambda\, t^\alpha) \text{ for } \alpha = \frac{1}{2} \text{ and } \arg(\lambda) = \pi/8$$

Real part



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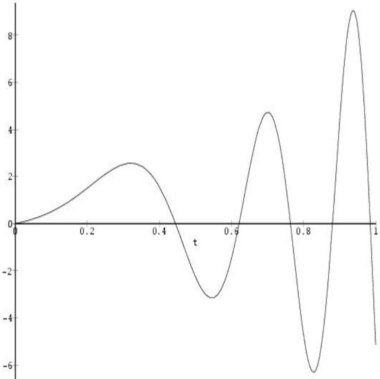
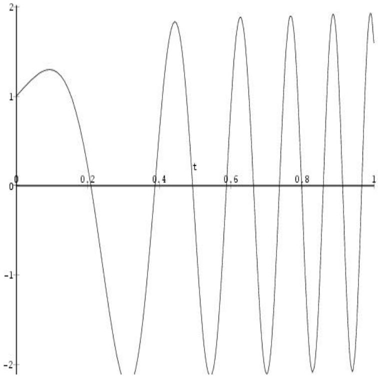


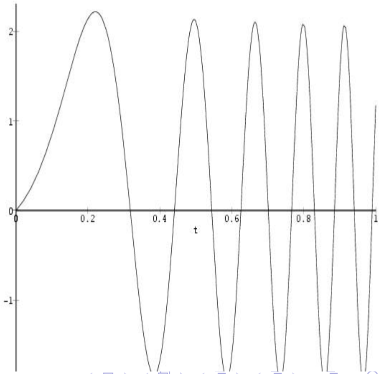
Mittag-Leffler functions in $\mathbb{C}$ (III)

$$t \mapsto E_\alpha(\lambda\, t^\alpha) \text{ for } \alpha = \frac{1}{2} \text{ and } \arg(\lambda) = \pi/4$$

Real part



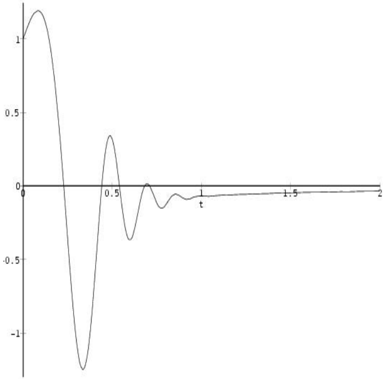
Imaginary part



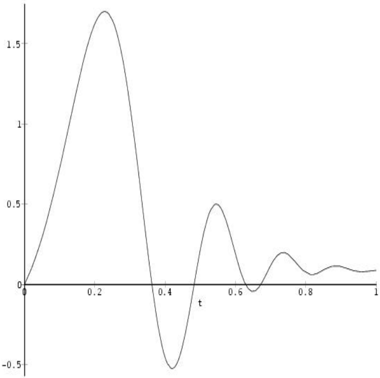
Mittag-Leffler functions in $\mathbb{C}$ (IV)

$$t \mapsto E_\alpha(\lambda\, t^\alpha) \text{ for } \alpha = \tfrac{1}{2} \text{ and } \arg(\lambda) = 3\pi/8$$

Real part



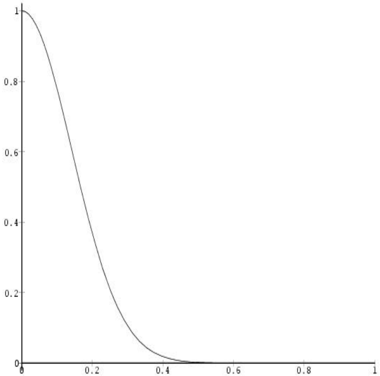
Imaginary part



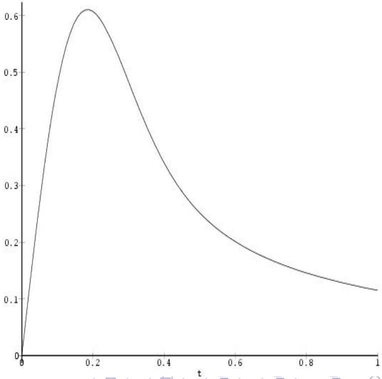
Mittag-Leffler functions in $\mathbb{C}$ (V)

$$t \mapsto E_\alpha(\lambda\, t^\alpha) \text{ for } \alpha = \tfrac{1}{2} \text{ and } \arg(\lambda) = \pi/2$$

Real part



Imaginary part



Mittag-Leffler functions in $\mathbb{C}$ (VI)

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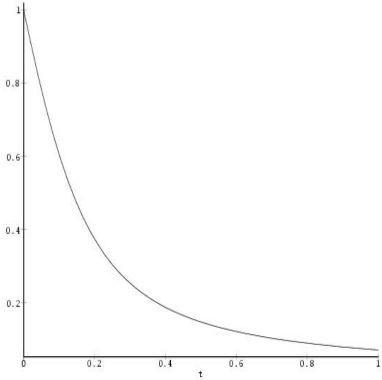
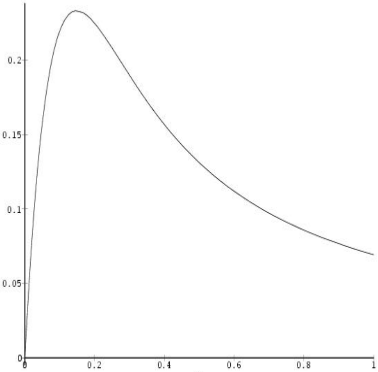
Real part



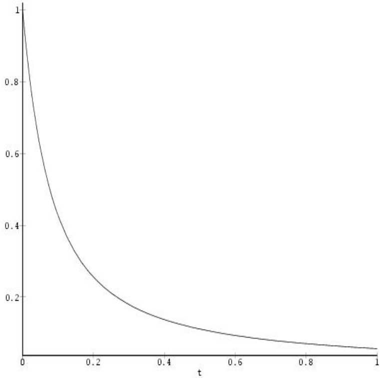
Imaginary part



Mittag-Leffler functions in $\mathbb{C}$ (VII)

$$t \mapsto E_\alpha(\lambda\, t^\alpha) \text{ for } \alpha = \frac{1}{2} \text{ and } \arg(\lambda) = \pi$$

Real part



Imaginary part

Outline

Webster-Lokshin fractional PDE

For $z \in (0,\,1)$, with $r(z) > 0$, $\eta(z), \varepsilon(z) \geq 0$, $w(t,z)$ satisfies :

$$\partial_t^2 w + \eta(z)\,\partial_t^{3/2} w + \varepsilon(z)\,\partial_t^{1/2} w - \frac{1}{r^2}\partial_z(r^2\,\partial_z w) = 0\,;$$

with static boundary conditions in $z = 0$ and $z = 1$.

- This is equivalent to the first-order system in $(p,\,v)$:

$$\begin{aligned} \partial_t p &= -r^{-2}\,\partial_z v - \varepsilon\,\partial_t^{-1/2} p - \eta\,\partial_t^{1/2} p\,, \\ \partial_t v &= -r^2\,\partial_z p\,, \\ p(z=0,t) &= 0 \quad \text{and} \quad v(z=1,t) = 0\,. \end{aligned}$$

- Use of standard DR for $\partial_t^{-1/2}$, and extended DR for $\partial_t^{1/2}$.

Standard DR : definitions

Let M a positive measureon $\mathbb{R}^+$ satisfying the well-posedness condition (WP) :

$$c_M \triangleq \int_0^\infty \frac{\mathrm{d}M}{1+\xi} < +\infty\,.$$

- We define the dynamical system with input $u \in L^2(0,T)$, output $y \in L^2(0,T)$ and state $\phi \in H_M = L^2(\mathbb{R}^+, \mathrm{d}M)$:

$$\begin{aligned} \partial_t\phi(\xi,t) &= -\xi\,\phi(\xi,t) + u(t); \quad \phi(\xi,0) = 0, \quad \forall\,\xi \in \mathbb{R}^+\,, \\ y(t) &= \int_0^{+\infty} \phi(\xi,t)\,\mathrm{d}M(\xi)\,. \end{aligned}$$

- Then, $y = h_M \star u$ where the impulse response can be written as $h_M(t) = \int_0^\infty e^{-\xi\,t}\,\mathrm{d}M(\xi)$ for $t > 0$.
- The transfer function is $\mathcal{H}_M(s) = \int_0^\infty \frac{\mathrm{d}M(\xi)}{s+\xi}$, in $\mathbb{C}_0^+$.

Standard DR : energy balance

The following energy balance is fulfilled, $\forall T > 0$:

$$\int_0^T u(t)\, y(t)\, dt = \frac{1}{2} \int_0^{+\infty} \phi(\xi, T)^2 \, \mathrm{d}M + \int_0^T \int_0^{+\infty} \xi \, \phi(\xi, t)^2 \, \mathrm{d}M \, \mathrm{d}t \,,$$

where the right-hand side can be decomposed into two parts :

- a *storage* function, evaluated at time T only, $E_\phi(T) := \frac{1}{2} \|\phi(T)\|^2_{H_M}$,
- a residual energy dissipated along the time interval $(0, T)$.

Example : $M_\beta(\mathrm{d}\xi) \triangleq \frac{\sin \beta\pi}{\pi} \, \xi^{-\beta} \, \mathrm{d}\xi$ for $0 < \Re e(\beta) < 1$ fulfills (WP), which gives rise to a *diagonal realization* of the *fractional integral operator* of order β, the transfer function of which is $\mathcal{H}_\beta(s) = s^{-\beta}$.

Note : standard DR belong to the class of *well-posed systems*.

Extended DR : definitions

Let N a positive measure on $\mathbb{R}^+$ satisfying the well-posedness condition (WP).

- We define the dynamical system with input $u \in H^1(0,T)$, output $z \in L^2(0,T)$ and state $\widetilde{\phi} \in \widetilde{H}_N = L^2(\mathbb{R}^+, \xi\, \mathrm{d}N)$:

$$\begin{aligned}\partial_t \widetilde{\phi}(\xi,t) &= -\xi\, \widetilde{\phi}(\xi,t) + u(t)\,; \quad \widetilde{\phi}(\xi,0) = 0 \quad \forall \xi \in \mathbb{R}^+\,, \\ z(t) &= \int_0^{+\infty} \partial_t \widetilde{\phi}(\xi,t)\, \mathrm{d}N(\xi) = \int_0^{+\infty} \left[u(t) - \xi\, \widetilde{\phi}(\xi,t)\right] \mathrm{d}N(\xi)\,.\end{aligned}$$

- Then, $z = \widetilde{h_N} \star u = \frac{d}{dt}(h_N \star u)$ with derivative *in the sense of distributions* : the impulse response is the distribution $\widetilde{h_N} = \frac{d}{dt} \int_0^\infty e^{-\xi\, t}\, \mathrm{d}N(\xi)$. (N.B. One can also write $z = h_N \star \frac{\mathrm{d}}{\mathrm{d}t} u$ in the sense of functions).
- The transfer function reads $\widetilde{H}_N(s) = s \int_0^\infty \frac{\mathrm{d}N(\xi)}{s+\xi}$, in $\mathbb{C}_0^+$.

Extended DR : energy balance

The following energy balance is fulfilled, $\forall T > 0$:

$$\int_0^T u(t)\, z(t)\, dt = \frac{1}{2}\int_0^{+\infty} \xi\, \widetilde{\phi}(\xi, T)^2\, \mathrm{d}N + \int_0^T \int_0^{+\infty} (u - \xi\, \widetilde{\phi})^2\, \mathrm{d}N\, \mathrm{d}t$$

where the right-hand side can be decomposed into two parts :

- a *storage* function, evaluated at time T only, $\widetilde{E}_{\widetilde{\phi}}(T) = \frac{1}{2}\, \|\widetilde{\phi}(T)\|^2_{\widetilde{H}_N}$,
- a residual energy dissipated along the time interval $(0, T)$.

Example : $N_\alpha(\mathrm{d}\xi) \triangleq M_{1-\alpha}(\mathrm{d}\xi) = \frac{\sin \alpha\pi}{\pi}\, \xi^{-(1-\alpha)}\, \mathrm{d}\xi$ for $0 < \Re e(\alpha) < 1$ fulfills (WP), which gives rise to a *diagonal realization* of the *fractional derivative operator* of order α, the transfer function of which is $\widetilde{\mathcal{H}}_\alpha(s) = s.s^{-(1-\alpha)} = s^{+\alpha}$.

Note : Extended DR do not belong to the class of *well-posed systems*, in general.

Existence et uniqueness (I)

With $L^2_p = \{p, \int_0^1 p^2 r^2\, dz < \infty\}$, $L^2_v = \{v, \int_0^1 v^2 r^{-2}\, dz < \infty\}$, and $\mathcal{H} = L^2_p \times L^2_v \times L^2(0,1; H_M; \varepsilon\, r^2\, dz) \times L^2(0,1; \widetilde{H}_N; \eta\, r^2\, dz)$, the system can be put in the abstract form $\partial_t X + \mathcal{A}\, X = 0$, where :

$$\mathcal{A}\begin{pmatrix} p \\ v \\ \varphi \\ \widetilde{\varphi} \end{pmatrix} = \begin{pmatrix} r^{-2}\,\partial_z v + \varepsilon\,\int_0^{+\infty} \varphi\, dM + \eta\,\int_0^{+\infty} [p - \xi\,\widetilde{\varphi}]\, dN \\ r^2\,\partial_z p \\ \xi\varphi - p \\ \xi\widetilde{\varphi} - p \end{pmatrix};$$

$$D(\mathcal{A}) = \left\{ (p, v, \varphi, \widetilde{\varphi})^T \in \mathcal{V}\,,\ \middle|\ \begin{array}{l} p(0) = 0 \\ v(1) = 0 \\ (p - \xi\varphi) \in L^2(0,1; H_M; \varepsilon\ r^2\, dz) \\ (p - \xi\widetilde{\varphi}) \in L^2(0,1; V_N; \eta\, r^2\, dz) \end{array} \right\}.$$

with $\mathcal{V} = H^1_p \times H^1_v \times L^2(0,1; V_M; \varepsilon\, r^2\, dz) \times L^2(0,1; \widetilde{H}_N; \eta\, r^2\, dz)$.

Existence et uniqueness (II)

Theorem

The operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is maximal monotone.

The *monotonicity* of $\mathcal{A}$ comes from the energy identity :
$\forall X \in D(\mathcal{A})$,

$$(\mathcal{A}X, X)_{\mathcal{H}} = \int_0^1 \|\varphi\|^2_{\widetilde{H}_M} \varepsilon\, r^2\, dz + \int_0^1 \|p - \xi\, \widetilde{\varphi}\|^2_{H_N} \eta\, r^2\, dz \geq 0\,.$$

Corollary

Hille-Yosida theorem enables to conclude to the existence and uniqueness of a strong solution for the original problem.

Note : in case of dynamical boundary conditions, the Kalman-Yakubovich-Popov lemma will be used to realize the ouput impedance, which is a positive real rational function of s.

Internal asymptotic stability

The proof is difficult. Why ?

- In fact, LaSalle's invariance principle requires, in infinite dimension, the hypothesis of precompactnes of the trajectories ; but this latter hypothesis cannot be checked a priori for diffusive realizations, since a diffusion equation in an unbounded domain is hidden behind them, and the canonical injection from $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$ is not compact (Rellich theorem does not apply).

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- The refined spectral analysis of the infinitesimal generator $-\mathcal{A}$ of the semigroup on the Hilbert state $\mathcal{H}$ enables to use the stability result by Arendt–Batty or Lyubich–Phong, et helps prove the result of *internal asymptotic stability* ; the proof is quite involved (Lax–Milgram theorem for the FDE, and Fredholm alternative for the FPDE).

Definition and Analysis of numerical schemes

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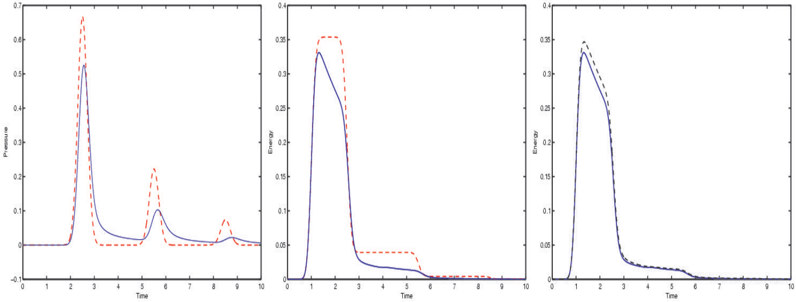
- Some Illustrations !

Influence of parameter η (I)

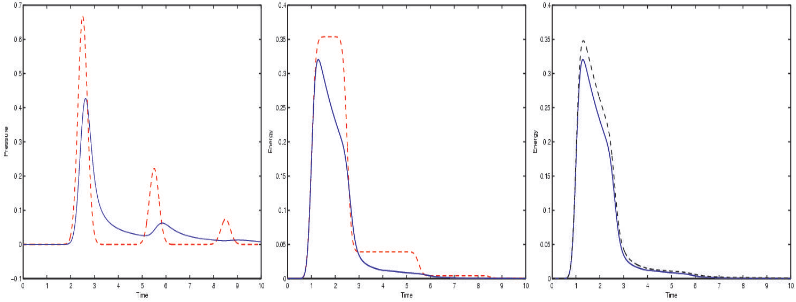
- Output signal. Wave & augmented (- -) energies



- For a cylinder, in blue $\eta = 0.1$, in red $\eta = 0$.

Influence of parameter η (II)

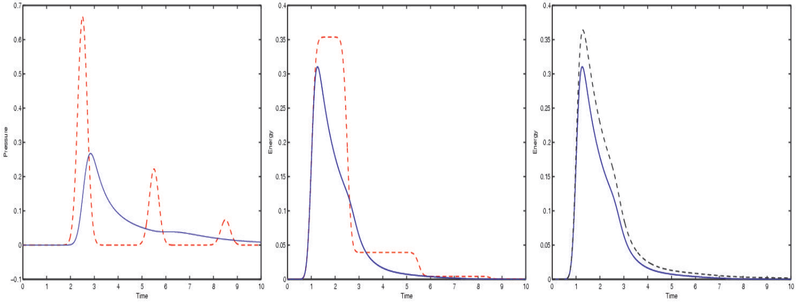
- Output signal. Wave & augmented (- -) energies.



- For a cylinder, in blue $\eta = 0.2$, in red $\eta = 0$.

Influence of parameter η (III)

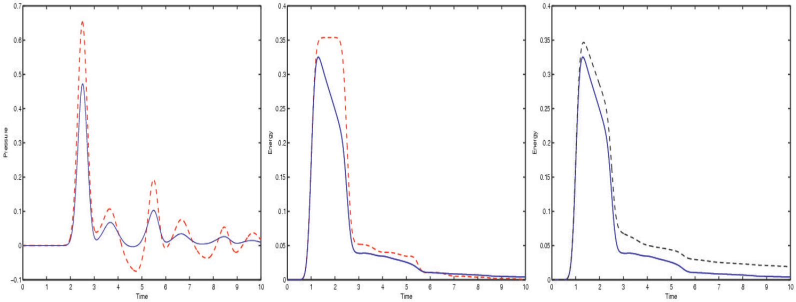
- Output signal. Wave & augmented (- -) energies.



- For a cylinder, in blue $\eta = 0.5$, in red $\eta = 0$.

Example of a trapped modes

- Output signal. Wave & augmented (- -) energies.



- Trapped modes in a duct with two cones facing each other, in blue $\varepsilon = 0.2$ and $\eta = 0.05$, in red $\varepsilon = \eta = 0$.

Outline

Un résultat général

Theorem

Soit H un espace de Hilbert, soit $A : D(A) \subset H \to H$ un opérateur maximal monotone, et F une fonction non-linéaire $F : H \to H$ telle que le pb. d'évolution semi-linéaire :

$$\partial_t X + A X = F(X), \quad \text{et } X(0) = X_0 \in D(A)$$

soit bien posé, pour $t \in [0, T_{max})$, au sens de l'existence et de l'unicité de $X \in C^1([0, T_{max}); H) \cap C^0([0, T_{max}); D(A))$, une solution forte.

Alors, pour deux OPD de type diffusif et positif, l'un standard $h_M\star$ et l'autre étendu par dérivation $\widetilde{h_N}\star$, le pb. pseudo-différentiel non-linéaire :

$$\partial_t X + h_M \star X + \widetilde{h_N} \star X + A X = F(X), \ \text{et } X(0) = X_0 \in D(A)$$

est bien posé, pour $t \in [0, T'_{max})$, au sens d'une unique solution forte $X \in C^1([0, T'_{max}); H) \cap C^0([0, T'_{max}); D(A))$.

Perturbation diffusive de systèmes non-linéaires conservatifs

Remark

L'hypothèse du théorème précdent est vérifiée dans le cas où la non-linéarité est localement lipschitzienne sur H.

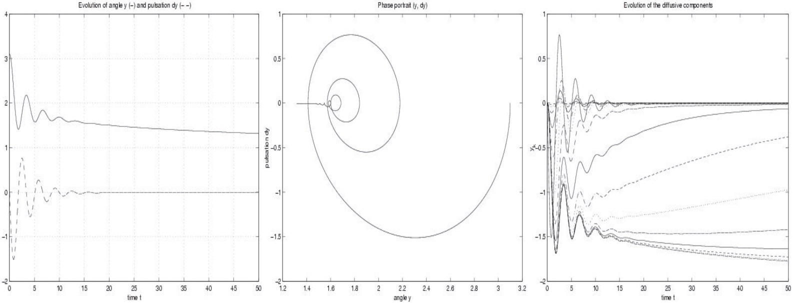
Corollary

Si le système différentiel non-linéaire de départ est conservatif, alors comme l'énergie étendue associée au système perturbé est décroissante et bornée par sa valeur initiale, le résultat d'existence locale se prolonge en un résultat d'existence globale, i.e. $T'_{max} = +\infty$.

Résultats de simulation (I) : pendule linéaire

- $\ddot{\vartheta} + \eta\, \partial_t^{-\beta} \dot{\vartheta} + \vartheta = 0$, pour $\beta = 0.75$ et $(\vartheta_0, \dot{\vartheta}_0) = (3.5, 0)$.

(G) (C) (D)

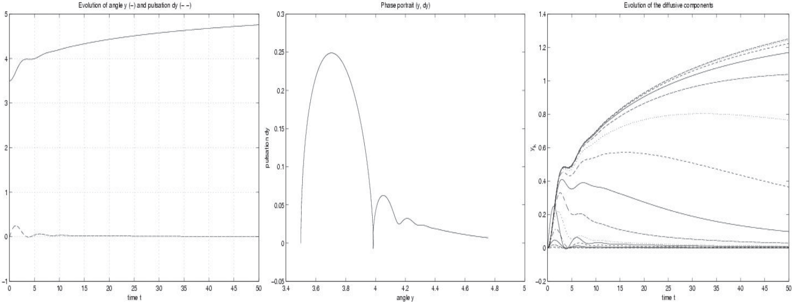


- (G) : Évolution de l'angle ϑ et de la vitesse angulaire $\dot{\vartheta}$,
  (C) : Section du portrait de phase dans le plan $(\vartheta, \dot{\vartheta})$,
  (D) : Évolution des composantes diffusives
  $\{\phi_k(t) = \phi(t, \xi_k)\}_{1 \le k \le K}$ pour $K = 25$.

Résultats de simulation (II) : pendule non-linéaire

- $\ddot{\vartheta} + \eta\, \partial_t^{-\beta} \dot{\vartheta} + \sin(\vartheta) = 0$; $\beta = 0.75$ et $(\vartheta_0, \dot{\vartheta}_0) = (3.5, 0)$.

(G) (C) (D)

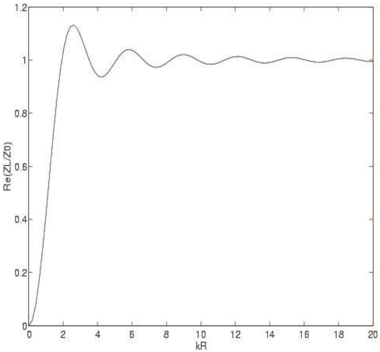


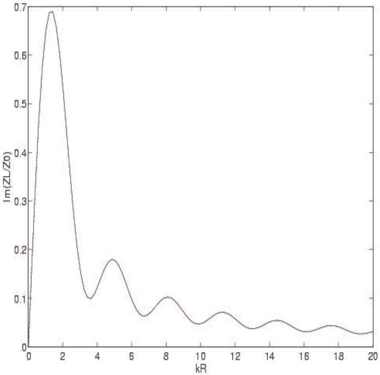
- (G) : Évolution de l'angle ϑ et de la vitesse angulaire $\dot{\vartheta}$,
  (C) : Section du portrait de phase dans le plan $(\vartheta, \dot{\vartheta})$,
  (D) : Évolution des composantes diffusives
  $\{\phi_k(t) = \phi(t, \xi_k)\}_{1 \leq k \leq K}$ pour $K = 25$.

An open question : Webster-Lokshin FPDE with Bessel-Struve radiation impedance ?

Is the following radiation impedance diffusive of the second kind ?

Bessel J_1 and *Struve* H_1 special functions





Real p. $x = kR \mapsto 1 - \frac{J_1(2x)}{2x}$ Imaginary p. $x = kR \mapsto \frac{H_1(2x)}{x}$

Matrix-valued impedances as diffusive PDOs ?

As an example, in order to solve the impedance matching problem for the Euler–Bernoulli beam, an impedance matrix of the form

$$\mathcal{Z} = \begin{bmatrix} a\,\partial_t^{+\alpha} & 1 \\ 1 & b\,\partial_t^{-\alpha} \end{bmatrix}$$

is found in the time domain, with $\alpha = \frac{1}{2}$.
A necessary and sufficient condition for the positivity of this operator is :

$$ab > \frac{1}{(\cos \alpha\pi)^2} \,.$$

Open question : is there any dissipative diffusive realization of the transfer matrix $\hat{\mathcal{Z}}(s)$, which is positive in the sense $\forall s \in \mathbb{C}_0^+$, $\hat{\mathcal{Z}}(s) + \hat{\mathcal{Z}}(s)^H \geq 0$, i.e. a positive symmetric real-valued matrix ?

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