Self-organized integrability in sytems with long range interactions

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Content of the Talk

Outline

System with Long range interactions

We consider classical Hamiltonian systems of interacting particles

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2} \sum_{i \neq j} V(q_i - q_j), \qquad (1)$$

with a two body interaction potential

$$V(r) \sim \frac{1}{r^\alpha}\,, \quad \alpha < d\,, \qquad (2)$$

where d is the dimension of space.

Examples

- Gravitation/Coulomb
  - Astronomy/astrophysics
  - Plasma physics
  - Wave particle interactions
- Fluid mechanics
  - System of point vortices
  - Stockeslets
- Mean Field/Toy models
  - HMF

Outline

Statistical and dynamical properties

Statistical properties

- Usually these systems are not extensive
  - Size dependent renormalization of the coupling constant
- Systems are not additive
  - Inequivalence between statistical ensembles is possible
  - Negative specific heat in the microcanonical ensemble

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Dynamical properties

- After a violent relaxation, systems sets in a long lived quasi-stationary state
  - Short time dynamics well described by the Vlasov equation
  - Slow relaxation towards equilibrium
  - Life of QSS scales with system size
- QSS states obtained through statistical physics using the Lynden-Bell formalism
  - Limits $t \to \infty$ and $N \to \infty$ do not commute

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The HMF model

The Hamiltonian of the HMF model is

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i \neq j} \cos(q_i - q_j). \qquad (3)$$

General properties

- $\alpha = 0 < d = 1$
- Coupling constant renormalized $1/N$: extensive
- System is not additive
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Statistical properties

- The magnetization is used as an order parameter

$$\mathbf{M} = \frac{1}{N} \left(\sum \cos q_i, \sum \sin q_i \right) = M \left(\cos \varphi, \sin \varphi \right) . \tag{4}$$

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Statistical properties

- The density of energy U becomes

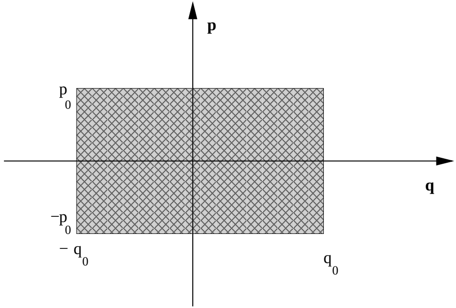
$$U = \frac{E}{N} = \frac{T}{2} - \frac{M^2}{2} . \quad (5)$$

- A second order phase transition exists at $U_c = 3/4$ and $T_c = 1/2$.

Outline

Out of equilibrium features

Out of equilibrium dynamics are computed using an initial water-bag:



This initial condition is then specified by $M_0 = \sin(q_0)/q_0$, and $U = p_0^2/6 + (1 - M_0^2)/2$.

Waterbag Dynamics

Mono-cluster

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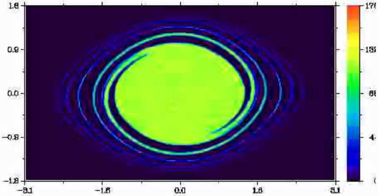

Bi-cluster

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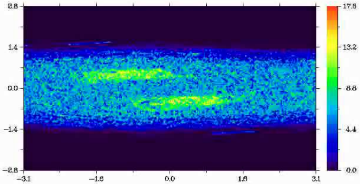
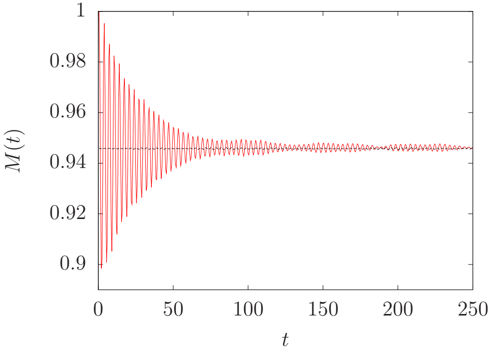


Poincaré "sections"



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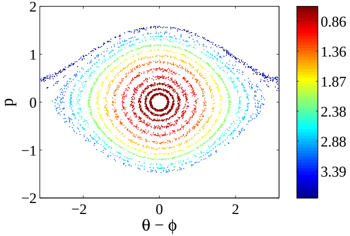

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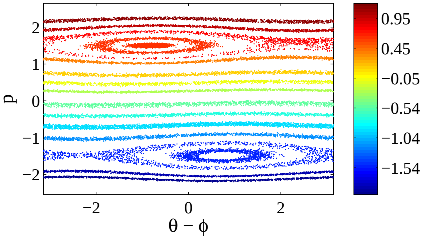
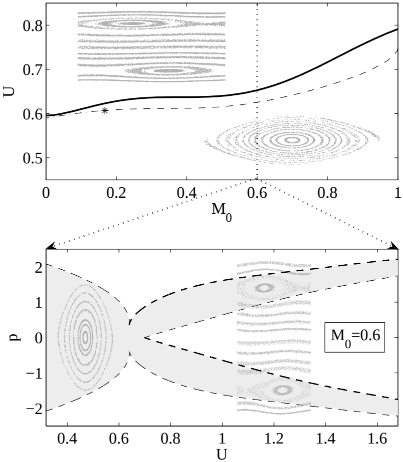


Out of equilibrium phase transition



Outline

A system of pendulum

The equation of motion are

$$\begin{cases} \dot{p}_i = -M \sin(q_i - \varphi) \\ \dot{q}_i = p_i \end{cases}, \tag{6}$$

Reminding us of a system of decoupled pendulum

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + m(1 - \cos q_i) , \tag{7}$$

Stationnary solutions

One pendulum

- $H = \frac{p^2}{2} + m(1 - \cos q)$,
- Compute an invariant ergodic measure

$$\rho_i(I, \theta) = \frac{1}{2\pi}\delta(I - I_i)$$

- for the collection of pendulum

$$\rho_E = \prod_{i=1}^{N} \rho_i \ , \qquad (8)$$

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Stationnary solutions

One particle PDF

We obtain the one particle PDF

$$f(I,\theta) = \frac{g(I)}{2\pi}\,, \Rightarrow\ \tilde{f}(p,q) \tag{9}$$

Stationary solution of the Vlasov equation of the pendulum system

Stationnary solutions

HMF Stationary State

- Recall

$$\left\{ \begin{array}{lcl} \dot{p}_i & = & -M \sin(q_i - \varphi) \\ \dot{q}_i & = & p_i \end{array} \right. , \tag{10}$$

- For the pendulum

$$\bar{\mathbf{M}} = \langle \mathbf{M} \rangle = \left(\frac{1}{2\pi} \int g(I) \cos q(I,\theta) \, dI d\theta \, , 0 \right) . \tag{11}$$

- And if $\langle \mathbf{M} \rangle = m$. Individual dynamics of pendulum and HMF systems are the same

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Example

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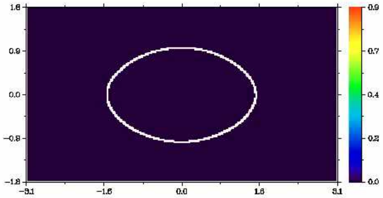


Outline

Lyapunov exponent

The one-particle distribution function at thermal equilibrium is

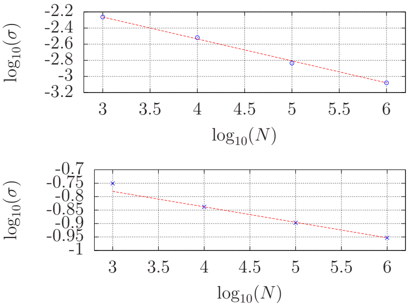
$$\rho(p,q) = \sqrt{\frac{2\pi}{\beta}} \frac{1}{I_0(\beta M)} \exp\left(-\beta\left(\frac{p^2}{2} - M\cos q\right)\right) , \qquad (12)$$

with M solution of the implicit equation

$$M = \frac{I_1(\beta M)}{I_0(\beta M)} \qquad (13)$$

Lyapunov exponent

We compute the Lyapunov exponent for finite N with close to equilibrium intital conditions:



Outline

The α-HMF model

The Hamiltonian of the HMF model is

$$H = \sum_{i=1}^{N} \left[\frac{p_i^2}{2} + \frac{1}{2\tilde{N}} \sum_{j \neq i}^{N} \frac{1 - \cos\left(q_i - q_j\right)}{\|i - j\|^{\alpha}} \right] , \qquad (14)$$

with

$$\tilde{N} = \left(\frac{2}{N}\right)^{\alpha} + 2 \sum_{i=1}^{N/2-1} \frac{1}{i^{\alpha}} \approx \frac{2}{1-\alpha} (N/2)^{1-\alpha}. \qquad (15)$$

General properties

- Long range for $0 \leq \alpha < d = 1$
- For $\alpha = 0$ the HMF model is recovered
- Equilibrium statistical properties are identical to HMF

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What about Stationary states

The equation of motion for p are

$$\dot{p}_i = -\sin(q_i)C_i + \cos(q_i)S_i = M_i \sin(q_i - \varphi_i)\,. \tag{16}$$

with

$$C_i = \frac{1}{\tilde{N}} \sum_{j \neq i} \frac{\cos q_j}{\|i-j\|^{\alpha}} \tag{17}$$

$$S_i = \frac{1}{\tilde{N}} \sum_{j \neq i} \frac{\sin q_j}{\|i-j\|^{\alpha}}\,, \tag{18}$$

and $M_i = \sqrt{C_i^2 + S_i^2}$.

What about Stationary states

Continuum limit

- We perform the $N \to \infty$,
- Using, $x = i/N$ and $y = j/N$ we obtain

$$C_i \approx \frac{1-\alpha}{2^\alpha} \sum_{j \neq i} \frac{1}{N} \frac{\cos q_j}{\|\frac{i}{N} - \frac{j}{N}\|^\alpha}$$

$$C_i \approx C(x) = \frac{1-\alpha}{2^\alpha} \int_{-1/2}^{1/2} \frac{\cos(q(y))}{\|x-y\|^\alpha} dy \,,$$

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What about Stationary states

Continuum limit

- We perform the $N \to \infty$
- And obtain thus scalar field equations

$$
\begin{aligned}
\frac{\partial q}{\partial t} &= p(x,t) \\
\frac{\partial p}{\partial t} &= \frac{\mu}{2^{\alpha}}\Gamma(\mu)\left(-\sin(q)I^{\mu}\left(\cos q\right) + \cos(q)I^{\mu}\left(\sin q\right)\right) ,
\end{aligned}
$$

with $\mu = 1 - \alpha$ and $I^{\mu}(f)$ represents a fractional integral

What about Stationary states

Fractional condition

So contrary to the mean field situation ($\alpha = 0$), the spatial organization $q(x)$ becomes relevant for $\alpha > 0$. We need $C(x) = Cte = \langle C \rangle = M$

$$\mathcal{D}^{\alpha} \cos q = \frac{d^{\alpha} \cos q}{dx^{\alpha}} = 0 \, . \tag{19}$$

HMF Stationary State

- Solution of the fractional equation imply:
- Spatial complex organization
  - Locally scale free distribution of values of $q(x)$
  - coarse grained value of $\cos q$ is constant.

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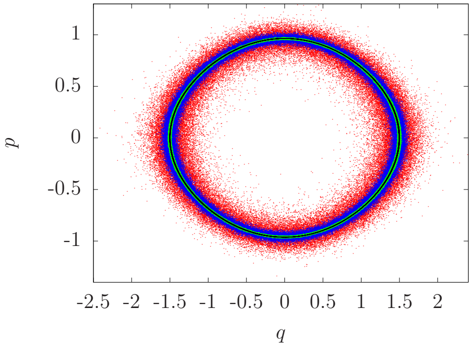

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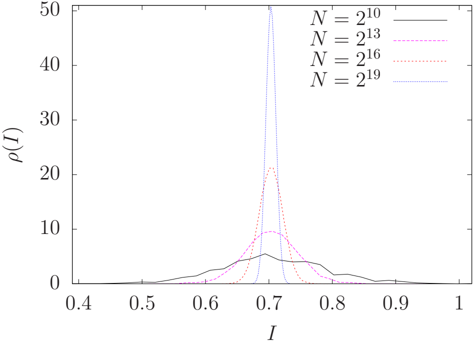
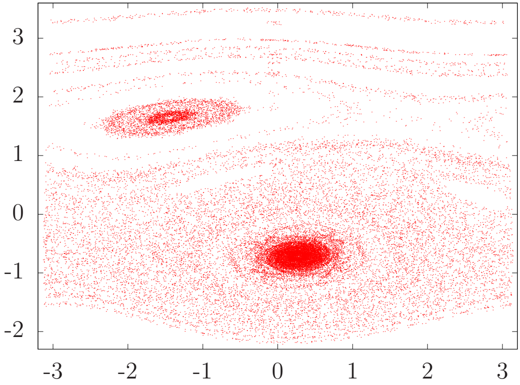


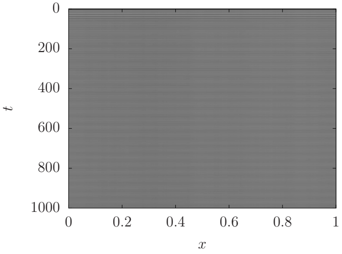
Poincaré “sections”

Multi-cluster



Poincaré "sections"

Structure of QSS



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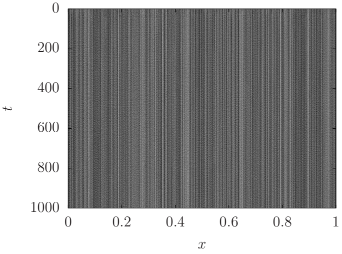
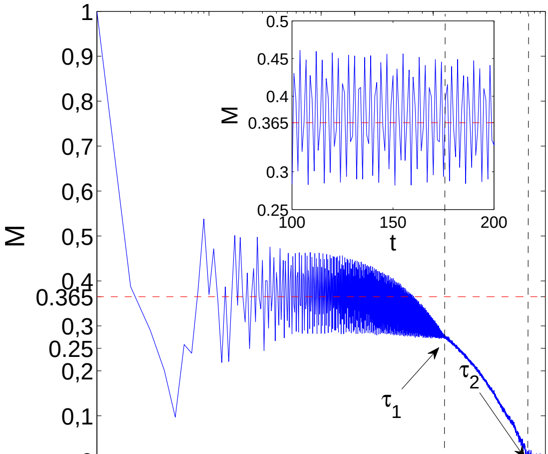


Lifetime

Magnetization vs time



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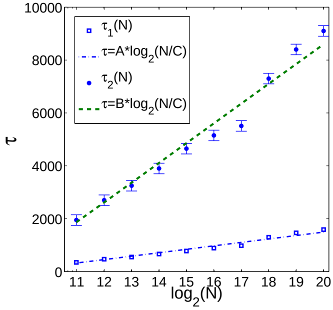
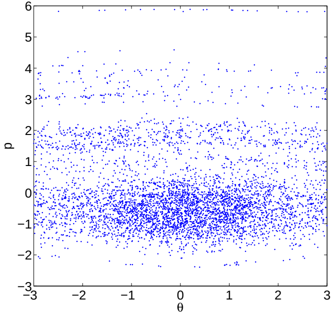


Structure of QSS



Conclusion

- The explored systems tend towards organizing themselves to have regular individual motion
- Stationary states correspond to integrable microscopic motion
- Regularity in time is preserved in non-mean field system at the price of spatial complexity
- Features of low-dimensional systems can be expected
- Fractional calculus appears to be useful
- Does it still hold for higher embeded dimensions and more realistic physical systems ?