Fractal Traps & Fractional Dynamics

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Dynamiques Fractionnaires et Applications, Pau

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General overview

- **Object:** Typical chaotic Hamiltonian systems
- **Aim:** Find causes of those fractional derivatives (in time)
- **Motivation:** Long-term behaviors of some Hamiltonian systems
- **Means:** Model based on studies of R. Hilfer and G.M. Zaslavsky

Outline

1. Fractional derivatives
2. Fractional dynamics & kinetics
3. Construction of a toy-model
4. Infinitesimal generators

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Fractional integral

- Fractional derivative: generalization of the usual derivative
- $$\int_{-\infty}^{t}\int_{-\infty}^{\tau_1}\cdots\int_{-\infty}^{\tau_{n-1}} f(\tau_n)\, d\tau_n \ldots d\tau_1 = \frac{1}{(n-1)!}\int_{-\infty}^{t}(t-\tau)^{n-1} f(\tau)\, d\tau$$
  $= \mathcal{I}_+^n f(t)$: n-th primitive of f
- Generalization to $\delta > 0$:
  $$\mathcal{I}_+^\delta f(t) = \frac{1}{\Gamma(\delta)}\int_{-\infty}^{t}(t-\tau)^{\delta-1} f(\tau)\, d\tau$$
  $\rightarrow$ fractional integral of order δ ($\Gamma(\delta)$: Gamma function)

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Fractional derivatives

- If $\omega \in (0,1)$, derivative of order ω:

$$\mathcal{D}_+^\omega f(t) = \frac{d}{dt}\mathcal{I}_+^{1-\omega} f(t) = \frac{1}{\Gamma(1-\omega)}\frac{d}{dt}\int_{-\infty}^{t}(t-\tau)^{-\omega} f(\tau)\, d\tau$$

$\rightarrow$ Liouville fractional derivative of order ω

- More convenient form (equivalent for sufficiently "good" functions):

$$\mathbf{D}_+^\omega f(t) = \frac{\omega}{\Gamma(1-\omega)}\int_0^{\infty}\tau^{-(1+\omega)}\left[f(t)-f(t-\tau)\right] d\tau$$

$\rightarrow$ Marchaud fractional derivative of order ω

- $\mathbf{D}_+^\omega f(t)$ exists if f bounded and locally ν-Hölderian, with $\nu > \omega$:

$$\forall t \in \mathbb{R},\ \exists c > 0,\ \exists \mu > 0,\ |\tau| \leq \mu \Rightarrow |f(t) - f(t-\tau)| \leq c|\tau|^\nu$$

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Applications

- Anomalous diffusion
- Memory effects
- Acoustics
- Porous media
- Phase transitions
- ...

Issues

Difficulties:

- Physical interpretation?
- Determination of ω?

$\rightarrow$ True signification or synthetic useful tool?

Some attempts to answer:

- R. Hilfer: fractional dynamics in dynamical systems
  $\rightarrow$ *interpretation*, ω
- G.M. Zaslavsky: fractional kinetics in chaotic Hamiltonian systems
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- Toy-model presented here: combination of the 2 approaches
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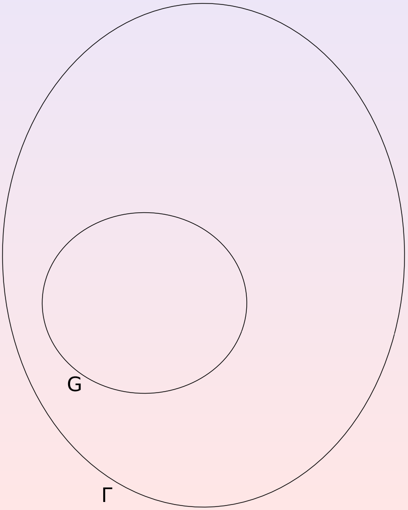
Fractional dynamics: induced dynamics

Dynamics in subsystems?



- Γ compact, $G \subset \Gamma$
- Flow $\phi^t x \rightarrow$ leaves G!
- Poincaré recurrence time:

$$\tau_G(x) = \inf\{k \geq 1 \mid \phi^{k\Delta t} x \in G\}$$

$\rightarrow$ Induced dynamics:

$$S(\Delta t)x = \phi^{\tau_G(x)} x$$

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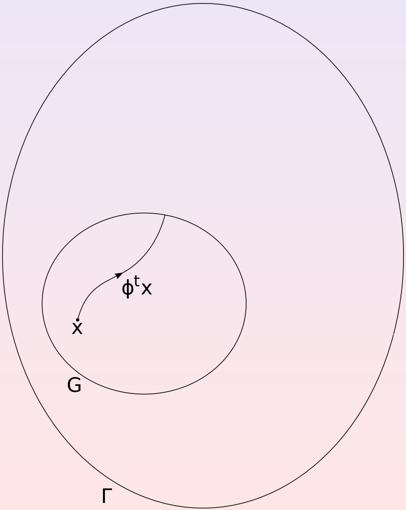

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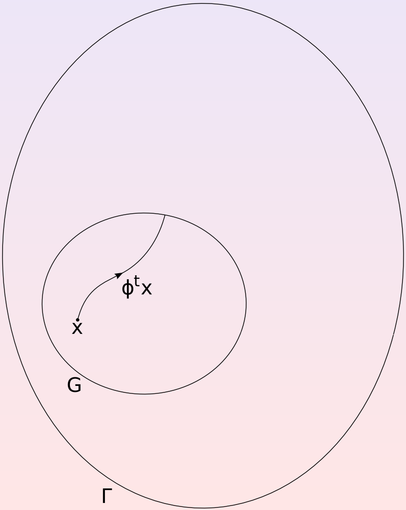

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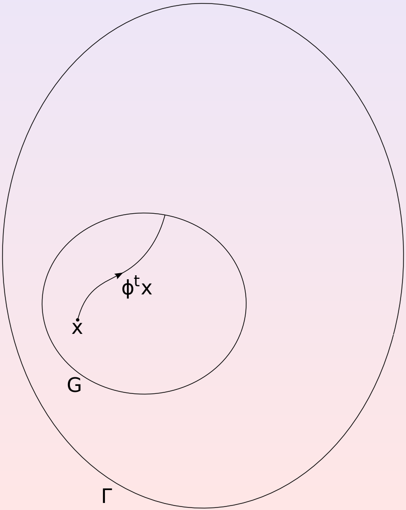

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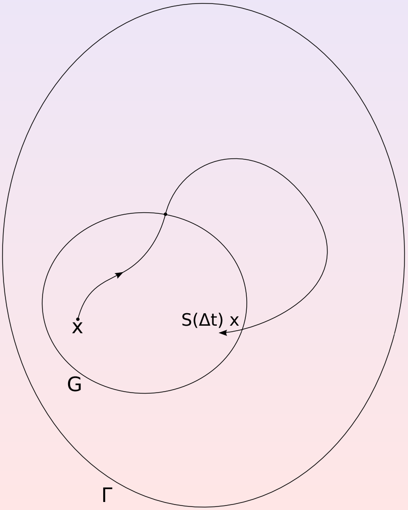

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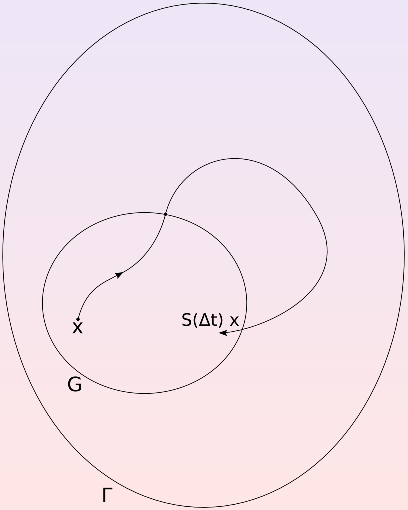

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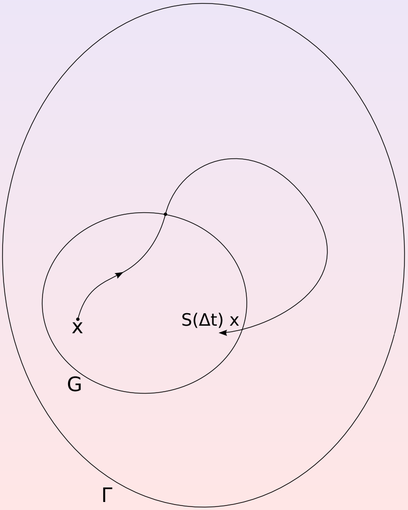

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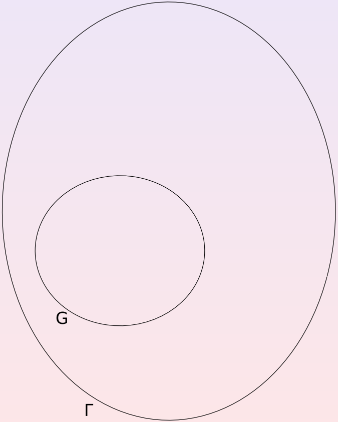

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- "Infinitesimal evolution":

$$S(\Delta t)\rho(B, t_0) = \sum_{k=1}^{\infty} p_k(\Delta t)\phi^{k\Delta t}\rho(B, t_0)$$
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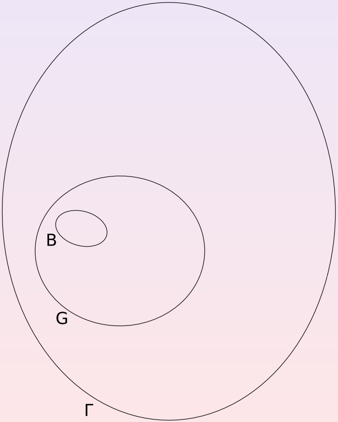

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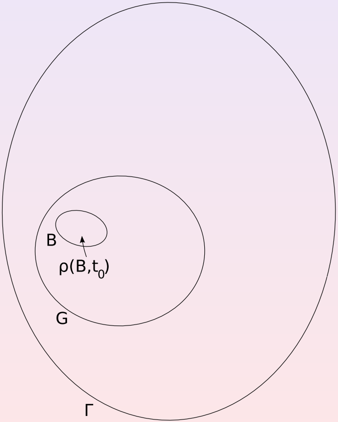

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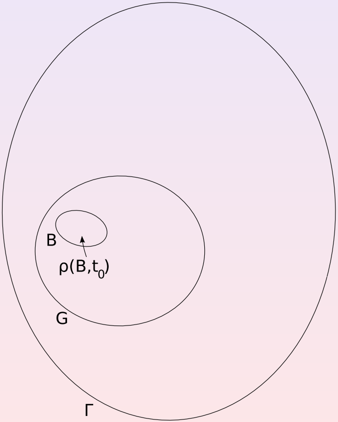

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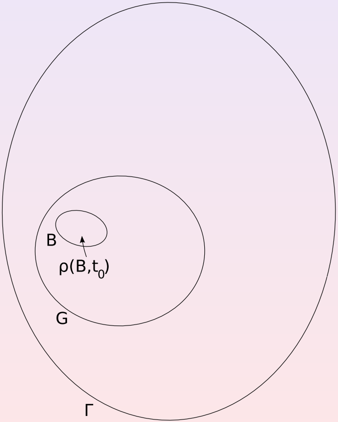

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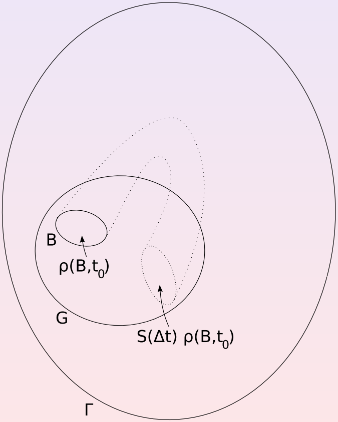

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Frational dynamics: infinitesimal generator

- Renormalization for long-time limit: $S(\Delta t) \rightarrow \tilde{S}(\widetilde{\Delta t})$
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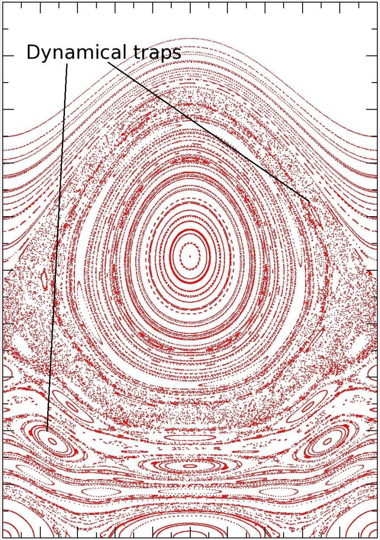
Fractional kinetics: fractal traps



- Typical chaotic Hamiltonian systems (with classical derivatives)
- Around kam tori, “dynamical traps”
- Fractal structure: $P_1 \supset P_2 \supset \ldots \supset P_n \ldots$

  $\rightarrow V_{n+1} = \lambda_S V_n, \quad \lambda_S < 1$
- Trapping times also auto-similar:

  $T_{n+1} = \lambda_T T_n, \quad \lambda_T > 1$

$\rightarrow$ "Sticky" zones

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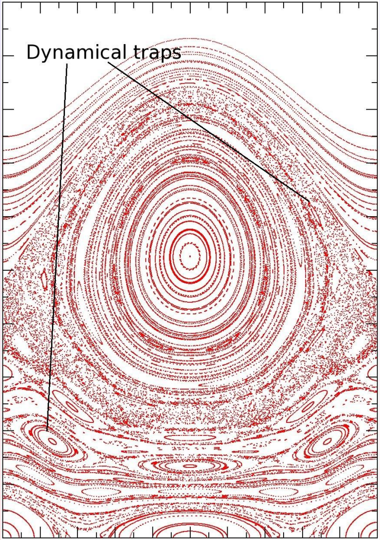

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- Trapping times also auto-similar:

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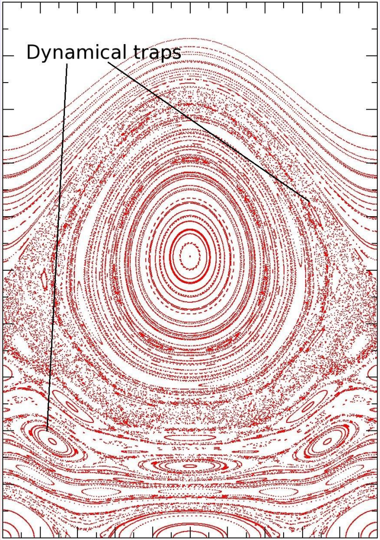
$\rightarrow$ "Sticky" zones

Fractional kinetics: fractal traps



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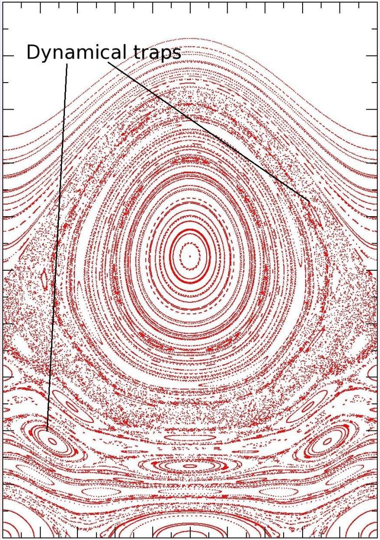

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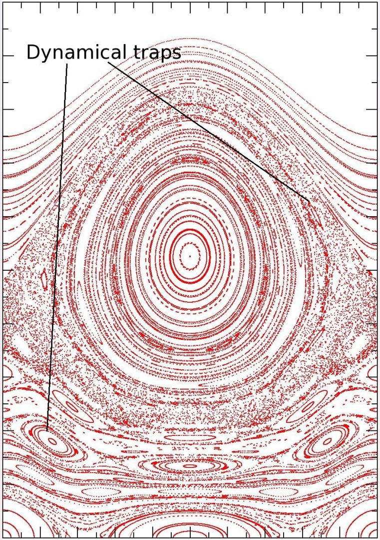

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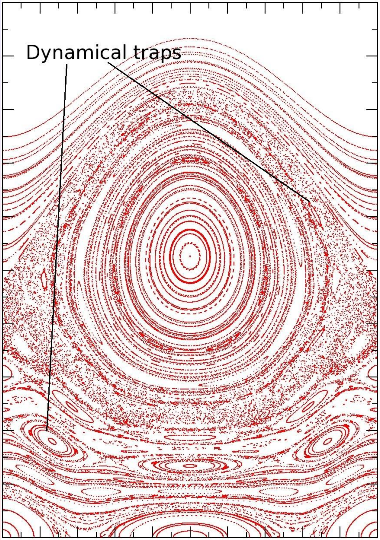

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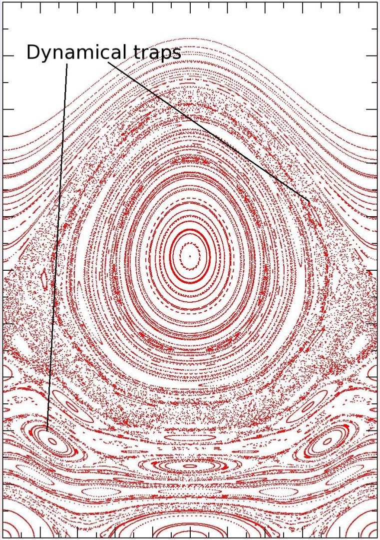

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Fractional kinetics: fractional equation

- Kinetic description:

$$\frac{\partial^\beta}{\partial t^\beta} P(x,t) = \frac{\partial^\alpha}{\partial x^\alpha}\left(\mathcal{A}(x)P(x,t)\right), \quad 0<\beta\leq 1,\ 0<\alpha\leq 2$$

- Anomalous diffusion: $\langle x^2\rangle \propto t^\mu$, with $\mu = \frac{2\beta}{\alpha}$: transport exponent
  - $0<\mu<1$: subdiffusion
  - $\mu=1$: normal diffusion
  - $\mu>1$: superdiffusion
- Origin: dynamical traps (?) $\rightarrow \mu = \frac{|\ln \lambda_S|}{\ln \lambda_T}$
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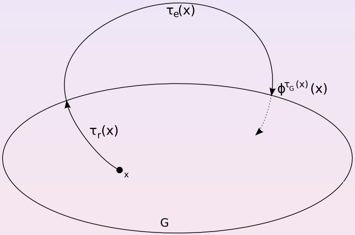
Outline

1. Fractional derivatives
2. Fractional dynamics & kinetics
3. Construction of a toy-model
4. Infinitesimal generators

General framework



- $G \subset \Gamma$, $0 < \nu(G) < \infty$
- $\tau_G(x) = \tau_r(x) + \tau_e(x)$
- Averaged dynamics restricted to G?
- $A \subset G$:

$$N_A : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ t & \longmapsto & \nu(\phi^t A \cap G) \end{array}$$

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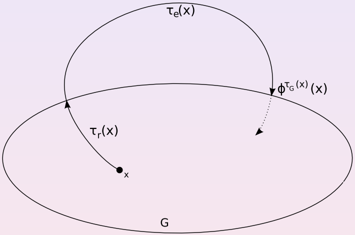

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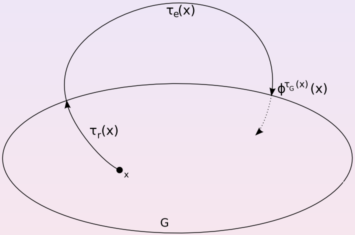

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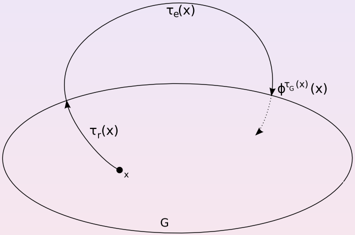

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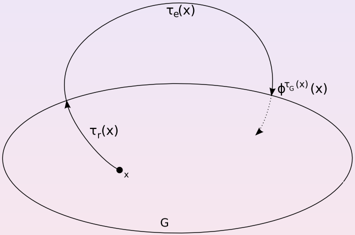

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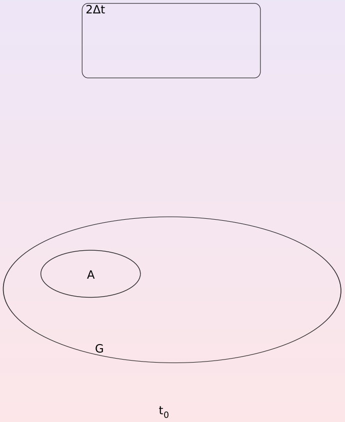

- Binary dynamics:
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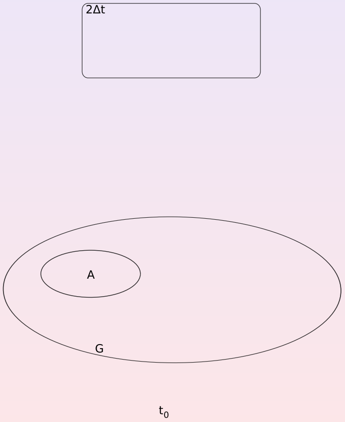

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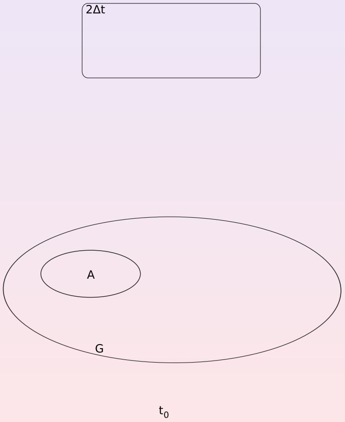

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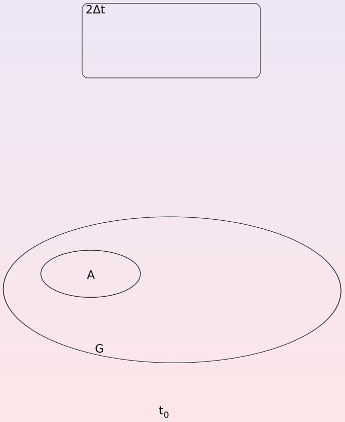

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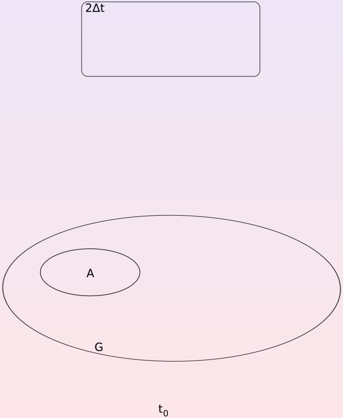

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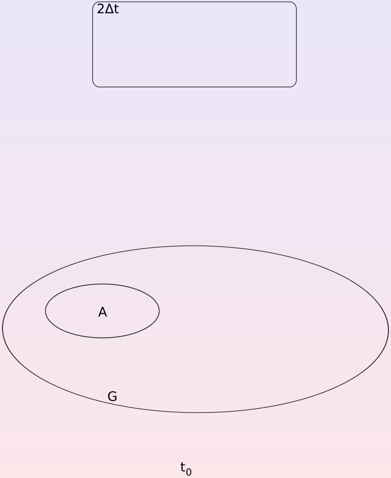

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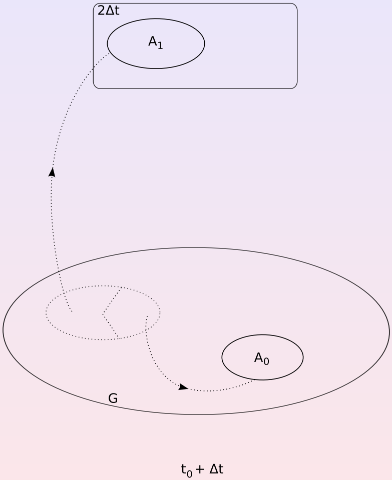

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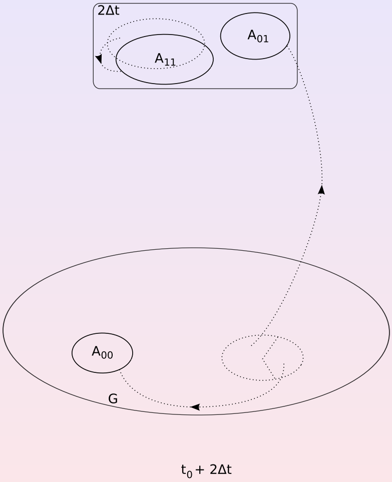

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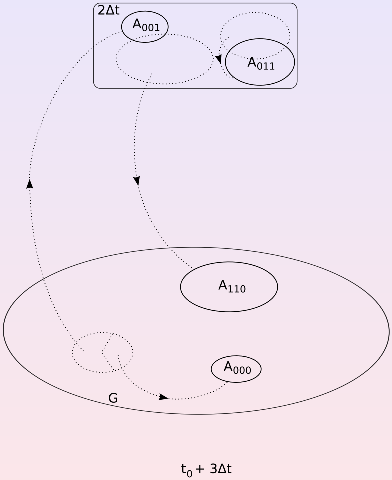

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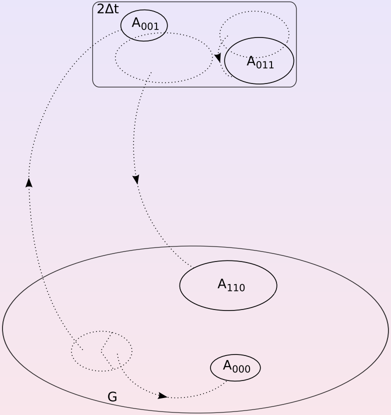

t0 + 3Δt

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- Infinitesimal evolution $S(\Delta t)N_A(t_0)$: "to the next step"
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- Traps $(P_k,\ T_k(\Delta t)),\ \dfrac{T_{k_1}}{T_{k_2}} \notin \mathbb{Q}$ $\rightarrow$ next step $\neq \Delta t$!

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- Zaslavsky: chaotic Hamiltonian system (with $\frac{d}{dt}$)

  → "averaged" dynamics?

- $V_{k+1} = \lambda_S(\Delta t) V_k$ and $T_{k+1} = \lambda_T(\Delta t) T_k$
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Outline

1. Fractional derivatives
2. Fractional dynamics & kinetics
3. Construction of a toy-model
4. Infinitesimal generators

Case $\mu > 1$

- Zaslavsky: $\mu = \dfrac{|\ln \lambda_S(\Delta t)|}{\ln \lambda_T(\Delta t)} = \dfrac{|\ln \lambda_s|}{\ln \lambda_t}$
- Assumption: $T_1(\Delta t) \underset{0^+}{\sim} a.\Delta t$

Theorem (Case $\mu > 1$)

If N_A is Lipschitz continuous and differentiable on $\mathbb{R}$, then

$$\mathcal{G}N_A(t_0) = -\gamma \frac{d}{dt} N_A(t_0),$$

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Theorem (Case $\mu < 1$)

If N_A satisfies the Hölder condition of order β and locally satisfies the Hölder condition of order δ, with $\beta < \mu < \delta$, then

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Fractional exponent

- Order of the derivative: explicitly determined
- $\frac{\partial^\beta}{\partial t^\beta} P(x,t) = \frac{\partial^\alpha}{\partial x^\alpha} \left(\mathcal{A}(x) P(x,t)\right)$ not explained with this model
- But for $\frac{\partial^\beta}{\partial t^\beta}$: "generalized shift of $P(x,t)$ along t by Δt" (Zasl.)
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- Combined approach based on R. Hilfer and G.M. Zaslavsky works
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