Vanishing results for cohomology
of complex toric hyperplane complements

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Toric arrangements

Let $T = (\mathbb{C}^*)^n$ be a complex torus and $\Lambda = Hom(T, \mathbb{C}^*) \simeq \mathbb{Z}^n$ the group of characters of T. Given an element $a \in \mathbb{C}^*$ and a character $\chi \in Hom(T, \mathbb{C}^*)$, the subtorus

$$H_{\chi,a} := \{t \in T \mid \chi(t) = a\}$$

is a *toric hyperplane* of T.

Then a finite subset $X \in \Lambda \times \mathbb{C}^*$ defines on T the *toric arrangement*:

$$\mathcal{T}_X := \{H_{\chi,a}, (\chi, a) \in X\}$$

and we get the *complement* of the arrangement:

$$\mathcal{R}_X := T \setminus \bigcup_{(\chi,a) \in X} H_{\chi,a}.$$

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Let $\pi : V \longrightarrow T$ be the universal covering of T.
Then V is a complex vector space of rank n, and π is the quotient map $\pi : V \longrightarrow V/\Lambda$, where Λ is a lattice in V.
Then the preimage $\pi^{-1}(H_{\chi,a})$ of a hypersurface $H_{\chi,a} \in \mathcal{T}_X$ is the union of an infinite family of parallel hyperplanes. Thus

$$\mathcal{A}_X := \{H \text{ hyperplane of } V \mid \exists(\chi, a) \in X \text{ s.t. } \pi(H) = H_{\chi,a}\}$$

is a periodic affine hyperplane arrangement in V with complement:

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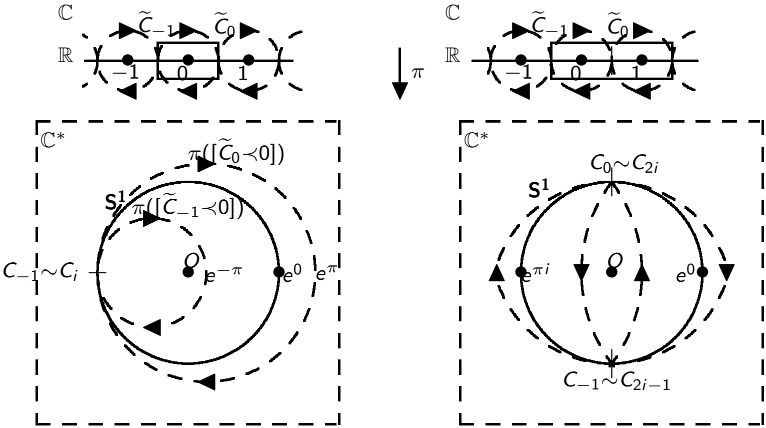

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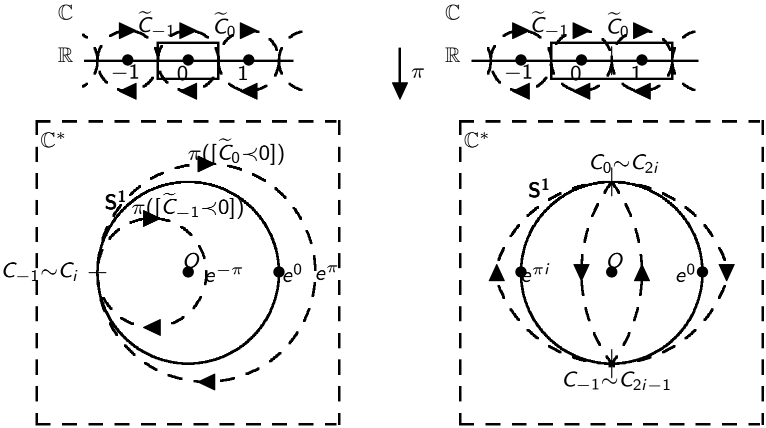

Vanishing results for hyperplane arrangements

Let A be a complex local system of coefficients, $\mathcal{A}$ an hyperplane arrangement in the complex space and $M(\mathcal{A})$ its complement. Then necessary conditions for $H^k(M(\mathcal{A}); A) = 0$ if $k \neq n$ have been determined by a number of authors, including Kohno , Esnault, Schechtman and Viehweg, Davis, Januszkiewicz and Leary, Schechtman, Terao and Varchenko and D. Cohen and Orlik .

In order to generalize some of the above results we use techniques developed by M. Davis in a joint work with Januszkiewicz, Leary and Okun, [1, 2].

Remark

In 2010, Papadima and Suciu generalize the result in the Cohen and Orlik paper to arbitrary minimal CW-complex. Recently D'Antonio and Delucchi proved that the complement of a toric arrangement is a minimal space. Hence the Papadima and Suciu result applies to toric arrangements.

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Certain cover and their nerves

Notations

- The *singular set*, Σ_X, is the union of toric hyperplanes in the arrangement.
- Its *complement*, $T - \Sigma_X$, is denoted $\mathcal{R}_X$.
- The *intersection set* L_X is the set of nonempty intersections of toric hyperplanes and $\overline{L}_X = L_X \cup \{T\}$. $\overline{L}_X$ is partially ordered by inclusion.
- The *rank* of the arrangement is the dimension of the linear subspace of $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $p(X) := \{\chi \mid (\chi, a) \in X\}$.
- The arrangement is *essential* if its rank is n.

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- The arrangement is *essential* if its rank is n.

Remark

Given a toric arrangement $\mathcal{T}_X$ of rank l, put $K_X := \bigcap_{\chi \in p(X)} \operatorname{Ker} \chi$ and $\overline{T}_X := T/K_X$. Thus, K_X and $\overline{T}_X$ are tori of dimensions $n-l$ and l, respectively. If $\overline{\Sigma}_X$ denotes the image of Σ_X in $\overline{T}_X$. We have a homeomorphism of pairs:

$$(T, \Sigma_X) \cong K_X \times (\overline{T}_X, \overline{\Sigma}_X) \quad (1)$$

and toric arrangement $\overline{\mathcal{T}}_X$ is essential (it is called the *essentialisation* of $\mathcal{T}_X$). So it is not restrictive to consider essential toric arrangements.

Small open subsets

Let U be a *convex subset* of T, i.e. a geodesically convex subset.

The intersection of an open convex subset of T with the toric hyperplanes in $\mathcal{T}_X$ is equivalent to an affine arrangement.

An open convex subset $U \subset T$ is *small* (with respect to $\mathcal{T}_X$) if this arrangement is central. In other words if the following two conditions hold (cf. [1]):

1. $\{G \in \overline{L}(\mathcal{T}_X) \mid G \cap U \neq \emptyset\}$ has a unique minimum element, $\mathrm{Min}(U)$.
2. A toric hyperplane $H \in \mathcal{T}_X$ has nonempty intersection with U if and only if $\mathrm{Min}(U) \subset H$.

Remark

If (i) and (ii) hold, then the arrangement in U is equivalent to the tangential central arrangement along $\mathrm{Min}(U)$ that we denote by $\mathcal{A}_{\mathrm{Min}(U)}$.

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$$\mathcal{U}_{\text{sing}} := \{U \in \mathcal{U} \mid U \cap \Sigma_X \neq \emptyset\}$$

and $\widehat{\mathcal{U}} = \{U - \Sigma_X\}_{U \in \mathcal{U}}$ the restriction to $\mathcal{R}_X$. Given a nonempty subset $\sigma \subset I$, put $U_\sigma := \bigcap_{i \in \sigma} U_i$.

Definition

The *nerve* $N(\mathcal{U})$ of $\mathcal{U}$ is the simplicial complex defined as follows. Its vertex set is I and a finite, nonempty subset $\sigma \subset I$ spans a simplex of $N(\mathcal{U})$ if and only if U_σ is nonempty.

Definition

$\beta(\mathcal{T}_X)$ is the rank of $H^n(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$.

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Generic local coefficients

Recall that, if $\mathcal{A}$ is a central arrangement, a rank one local system is *generic* if the $\mathcal{A}$-tuple of coefficients $\lambda_H \in \mathbb{C}^*$ associated to the loops a_H around hyperplanes $H \in \mathcal{A}$ satisfies

$$\prod_{H \in \mathcal{A}} \lambda_H \neq 1.$$

A local coefficient system $A_{\Lambda_T} \in \mathrm{Hom}(H_1(\mathcal{R}_X), \mathbb{C}^*)$ on $\mathcal{R}_X$ is generic if the localization of A_{Λ_T} on any open set $\widehat{U}_\sigma$ is generic.

ℓ^2 and group ring coefficients

ℓ^2 and group ring cohomologies are defined in the standard way.

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Statements of the main theorems

Suppose $\mathcal{T}_X$ is an essential toric arrangement in T and $\pi = \pi_1(\mathcal{R}_X)$.

Theorem

Let Λ_T a generic X-tuple with entries in k^*. Then $H^*(\mathcal{R}_X; A_{\Lambda_T})$ is concentrated in degree n and

$$\dim_k H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(\mathcal{T}_X).$$

Theorem

The ℓ^2-Betti numbers of $\mathcal{R}_X$ are 0 except in degree n and $\ell^2 b_n(\mathcal{R}_X) = \beta(\mathcal{T}_X)$.

Theorem

(cf. [1, 2]). *$H^*(\mathcal{R}_X; \mathbb{Z}\pi)$ vanishes except in degree n, i.e. $\mathcal{R}_X$ is a duality space, and $H^n(\mathcal{R}_X; \mathbb{Z}\pi)$ is free abelian.*

Main ingredients of the proof

- Vanishing results for central arrangements in the complex space.
- Key Lemma
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Vanishing results for central arrangements in the complex space

Lemma

Suppose $\mathcal{A}$ is a finite, central arrangement of hyperplanes. Let $\pi' = \pi_1(M(\mathcal{A}))$. Then

1. *For any generic system of local coefficients A, $H^*(M(\mathcal{A}); A)$ vanishes in all degrees.*
2. *$H^*(M(\mathcal{A}); \mathcal{N}\pi')$ vanishes in all degrees. Hence, all ℓ^2-Betti numbers are 0.*
3. *If the rank of $\mathcal{A}$ is l, then $H^*(M(\mathcal{A}); \mathbb{Z}\pi')$ vanishes except in the top degree, l.*

Key Lemma

Lemma

Suppose $\mathcal{T}_X$ is essential. $N(\mathcal{U})$ is homotopy equivalent to T and $N(\mathcal{U}_{\text{sing}})$ is a subcomplex homotopy equivalent to Σ_X. Moreover, $H^*(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$ is concentrated in degree n and $H^n(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$ is free abelian.

Consequence

$\beta(\mathcal{T}_X)$ is the rank of $H^n(T, \Sigma_X)$ Also, it is not difficult to see that $(-1)^l\beta(\mathcal{T}_X) = e(T, \Sigma_X) = 1 - e(\Sigma_X) = e(\mathcal{R}_X)$, where $e(\,)$ denotes Euler characteristic.

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The proof of the above Lemma is a consequence of the following one.

Lemma

Suppose $\mathcal{T}_X$ is an essential toric arrangement on T and $\Sigma = \Sigma_X$. Then $H^*(T, \Sigma)$ is free abelian and concentrated in degree n.

Proof

The proof essentially follows a classical *deletion-restriction* argument using induction on $\mathrm{Card}(\mathcal{T}_X)$.

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The Mayer-Vietoris spectral sequence

Let $\mathcal{U} = \{U_i\}$ be an open cover of T by small convex sets and $\widehat{\mathcal{U}} = \{U - \Sigma_X\}_{U \in \mathcal{U}}$ the restriction to $\mathcal{R}_X$.

Open cover in $G \in \overline{L}_X$

For each $G \in \overline{L}_X$, put

$$\mathcal{U}_G := \{U \in \mathcal{U} \mid \mathrm{Min}(U) \leq G\},$$
$$\mathcal{U}_G^{\mathrm{sing}} := \{U \in \mathcal{U} \mid \mathrm{Min}(U) < G\} = \{U \in \mathcal{U}_G \mid U \cap \Sigma_X \cap G \neq \emptyset\}.$$

Any element $\widehat{U} = U - \Sigma_X$ of the cover is homotopy equivalent to the complement of a central arrangement $M(\mathcal{A}_{\mathrm{Min}(U)})$ and

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Let $r : \widetilde{\mathcal{R}}_X \to \mathcal{R}_X$ be the universal cover and $\pi = \pi_1(\mathcal{R}_X)$. The induced open cover $\{r^{-1}(\widehat{U})\}$ of $\widetilde{\mathcal{R}}_X$ has the same nerve $N(\widehat{\mathcal{U}})$ $(= N(\mathcal{U}))$.

We have the Mayer–Vietoris double complex,

$$C_{i,j} := \bigoplus_{\sigma \in N^{(i)}} C_j(r^{-1}(\widehat{U}_\sigma)),$$

where $N^{(i)}$ denotes the set of i-simplices in $N(\mathcal{U})$.

We get a corresponding double cochain complex,

$$E_0^{i,j} := \mathrm{Hom}_\pi(C_{i,j}, A), \tag{3}$$

The filtration on the double complex gives a spectral sequence converging to the associated graded module for cohomology:

$$\mathrm{Gr}\, H^m(\mathcal{R}_X; A) = E_\infty := \bigoplus_{i+j=m} E_\infty^{i,j}.$$

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Hence for any coface σ' of σ, if $G' := \mathrm{Min}(U_{\sigma'}) < G$, the coefficient homomorphism $H^j(M(\mathcal{A}_G); A) \to H^j(M(\mathcal{A}_{G'}); A)$ is the zero map and the E_1 page of the spectral sequence decomposes as

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E_2 page

$$E_2^{i,j} = \bigoplus_{G \in \overline{L}_X^{n-j}} H^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\mathrm{sing}}); H^j(M(\mathcal{A}_G); A))$$
$$= \bigoplus_{G \in \overline{L}_X^{n-j}} H^i(G, \Sigma_X \cap G; H^j(M(\mathcal{A}_G); A)), \qquad (4)$$

$A = A_{\Lambda_T}$ or $A = \mathcal{N}\pi$ cases

In these cases all summands vanish for $G \neq T$ and $j \neq 0$. We are left with $E_2^{n,0} = H^n(T, \Sigma_X; A)$ isomorphic to the tensor product free abelian group of rank $\beta(\mathcal{T}_X)$ with A.
This proves the first two Theorems.

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$A = \mathbb{Z}\pi$ case

In this case $H^i(G, \Sigma_X \cap G)$ is concentrated in degree $\dim G = n - j$. Hence, $E_2^{i,j}$ is nonzero (and free abelian) only for $i + j = n$. It follows that the spectral sequence degenerates at E_2, i.e., $E_2 = E_\infty$. This proves the last Theorem.

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