What is a graphical toric arrangement?

Arrangements in Pyrénées

Luca Moci

Marie Curie fellow of INdAM at Université de Paris 7

June 12, 2012
Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges.

A $q$–coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \forall (i, j) \in E$.

FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \quad \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.

FACT: the number $\chi^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.

What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x, y)$. 

Luca Moci (Marie Curie fellow of INdAM at Universitè de Paris 7)
Chromatic polynomial and flow polynomial

Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \rightarrow \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \quad \forall (i, j) \in E$. FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \rightarrow \mathbb{Z}/q$ such that

\[
\text{for every } v \in V, \quad \sum_{s(e) = v} f(e) = \sum_{t(e) = v} f(e).
\]

It is nowhere zero if $f(e) \neq 0 \quad \forall e \in E$.

FACT: the number $\chi^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.

What is most general deletion-contraction invariant of a graph?

The Tutte polynomial $T_G(x, y)$. 

Luca Moci (Marie Curie fellow of INdAM at Université de Paris 7)
Chromatic polynomial and flow polynomial

Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \rightarrow \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \forall (i,j) \in E$.

FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \rightarrow \mathbb{Z}/q$ such that

$$\sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.

FACT: the number $\chi^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x, y)$. 
Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \forall (i, j) \in E$. FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \quad \sum_{s(e) = v} f(e) = \sum_{t(e) = v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$. FACT: the number $\chi^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.

What is most general deletion-contraction invariant of a graph? The Tutte polynomial $T_G(x, y)$. 

Chromatic polynomial and flow polynomial

Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \rightarrow \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \ \forall (i, j) \in E$. FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \rightarrow \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \quad \sum_{s(e) = v} f(e) = \sum_{t(e) = v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.

FACT: the number $\chi^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.

What is most general deletion-contraction invariant of a graph?

The Tutte polynomial $T_G(x, y)$.
Chromatic polynomial and flow polynomial

Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \rightarrow \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \ \forall (i,j) \in E$. FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi_G(q) = \chi_{G \setminus e}(q) - \chi_{G / e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \rightarrow \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e) = v} f(e) = \sum_{t(e) = v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \ \forall e \in E$.

FACT: the number $\chi^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.

What is most general deletion-contraction invariant of a graph?

The Tutte polynomial $T_G(x, y)$. 

---

Luca Moci (Marie Curie fellow of INdAM at Università degli Studi di Roma Tor Vergata)
Chromatic polynomial and flow polynomial

Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \ \forall (i,j) \in E$. FACT: the number $\chi _G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi _G(q) = \chi _{G\setminus e}(q) - \chi _{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \quad \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \ \forall e \in E$.

FACT: the number $\chi ^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.

What is most general deletion-contraction invariant of a graph?

The Tutte polynomial $T_G(x,y)$. 

Luca Moci (Marie Curie fellow of INdAM at I)
Chromatic polynomial and flow polynomial

Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \rightarrow \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \forall (i, j) \in E$. FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \rightarrow \mathbb{Z}/q$ such that

for every $v \in V$, $\sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e)$.

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.

FACT: the number $\chi^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.

What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x, y)$. 

Luca Moci (Marie Curie fellow of INdAM at Università di Roma Tor Vergata)
Chromatic polynomial and flow polynomial

Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \forall (i, j) \in E$. FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi_G(q) = \chi_{G \setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e) = v} f(e) = \sum_{t(e) = v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.

FACT: the number $\chi^{*}_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.

What is most general deletion-contraction invariant of a graph?

The Tutte polynomial $T_G(x, y)$. 

Chromatic polynomial and flow polynomial

Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \forall (i, j) \in E$. FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$. FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \quad \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$. FACT: the number $\chi^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$. Again, we have deletion-contraction. What is most general deletion-contraction invariant of a graph? The Tutte polynomial $T_G(x, y)$. 

Luca Moci (Marie Curie fellow of INdAM at Università di Roma 1)
Chromatic polynomial and flow polynomial

Let $G$ be a graph, $V$ the set of vertices, $E$ the set of edges. A $q$–coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \forall (i, j) \in E$. FACT: the number $\chi_G(q)$ of proper $q$–colorings of $G$ is a polynomial in $q$, which we call the chromatic polynomial of $G$.

FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of $G$, a $q$–flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \quad \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$. FACT: the number $\chi^*_G(q)$ of nowhere zero $q$–flows of $G$ is a polynomial in $q$, which we call the flow polynomial of $G$.

Again, we have deletion-contraction.

What is most general deletion-contraction invariant of a graph? The Tutte polynomial $T_G(x, y)$. 

Luca Moci (Marie Curie fellow of INdAM at Université de Paris 7)
Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$. Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Möbius function $\mu$ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \hat{0}) q^{\dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).
The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$. Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Moebius function $\mu$ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \hat{0}) q^{\dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).
The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$. Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Möbius function $\mu$ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \hat{0}) q^{\dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).
The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$. Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Möbius function $\mu$ and a characteristic polynomial

$$\chi_\mathcal{H}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \hat{0}) q^{\dim(L)}.$$ 

The polynomial $\chi_\mathcal{H}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).
The characteristic polynomial

Let \( \mathcal{H} \) be a hyperplane arrangement in \( \mathbb{K}^n \). Its intersection lattice \( L(\mathcal{H}) \) is a ranked poset, hence it has a M"obius function \( \mu \) and a characteristic polynomial

\[
\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \hat{0}) q^{\dim(L)}.
\]

The polynomial \( \chi_{\mathcal{H}}(q) \) contains much information on the complement of the arrangement \( \mathcal{H} \), such as:

- the number of chambers and bounded chambers (if \( \mathbb{K} = \mathbb{R} \));
- the Betti numbers of the complement (if \( \mathbb{K} = \mathbb{C} \));
- the number of points in the complement (if \( \mathbb{K} = \mathbb{F}_q \)).
The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$.
Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Möbius function $\mu$ and a characteristic polynomial

$$
\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \hat{0}) q^{\dim(L)}.
$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).
Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$. Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Möbius function $\mu$ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \hat{0}) q^{\dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).
Let $G = (V, E)$ be a graph and $n = |V|$.

To $G$ we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.

**FACT:** $\chi_{\mathcal{H}(G)} = \chi_G$.

We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$.

Moreover, if $G$ is planar, it exists $G^*$ such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.

So the question arises naturally:

**Problem**

*What is a graphical toric arrangement?*
Graphical hyperplane arrangement

Let $G = (V, E)$ be a graph and $n = |V|$.
To $G$ we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $K^n$.

FACT: $\chi_{\mathcal{H}(G)} = \chi_G$.

We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$.
Moreover, if $G$ is planar, it exists $G^*$ such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.
So the question arises naturally:

Problem

What is a graphical toric arrangement?
Let $G = (V, E)$ be a graph and $n = |V|$.
To $G$ we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.

**FACT:** $\chi_{\mathcal{H}(G)} = \chi_G$.

We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$.
Moreover, if $G$ is planar, it exists $G^*$ such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.
So the question arises naturally:

**Problem**

*What is a graphical toric arrangement?*
Let $G = (V, E)$ be a graph and $n = |V|$. To $G$ we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.

**FACT:** $\chi_{\mathcal{H}(G)} = \chi_G$.

We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$. Moreover, if $G$ is planar, it exists $G^*$ such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively. So the question arises naturally:

**Problem**

*What is a graphical toric arrangement?*
Let $G = (V, E)$ be a graph and $n = |V|$. To $G$ we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $K^n$.

**FACT:** $\chi_{\mathcal{H}(G)} = \chi_G$.

We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi_*^G$. Moreover, if $G$ is planar, it exists $G^*$ such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively. So the question arises naturally:

**Problem**

What is a graphical toric arrangement?
Let $G = (V, E)$ be a graph and $n = |V|$. To $G$ we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.

**FACT:** $\chi_{\mathcal{H}(G)} = \chi_G$.
We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$.
Moreover, if $G$ is planar, it exists $G^*$ such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively. So the question arises naturally:

**Problem**

*What is a graphical toric arrangement?*
Let $G = (V, E)$ be a graph and $n = |V|$. To $G$ we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.

FACT: $\chi_{\mathcal{H}(G)} = \chi_G$.

We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$. Moreover, if $G$ is planar, it exists $G^*$ such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively. So the question arises naturally:

**Problem**

*What is a graphical toric arrangement?*
A list $X$ of elements of a finitely generated abelian group $\Gamma$ defines a \textit{(generalized) toric arrangement} $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $\text{Hom}(G, S^1)$, composed by a subgroup $H_e$ for every $e \in X$ (having codimension 0 if $e$ is torsion and 1 otherwise).

Example: \(\Gamma = \mathbb{Z}^2\), \(T = \text{Hom}(\Gamma, S^1) = (S^1)^2\).

The list $X = \{(2, 0), (0, 3), (-1, 1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given $T$ by the equations: \(t^2 = 1, s^3 = 1, t^{-1}s = 1\).

The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list $X$ is given in $\mathbb{R}^2$ by equations: \(2x = 0, 3y = 0, -x + y = 0\).

If we replace $(0, 3)$ by $(0, 5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$. 
Toric arrangements

A list $X$ of elements of a finitely generated abelian group $\Gamma$ defines a (generalized) toric arrangement $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $Hom(G, S^1)$, composed by a subgroup $H_e$ for every $e \in X$ (having codimension 0 if $e$ is torsion and 1 otherwise).

Example: $\Gamma = \mathbb{Z}^2$, $T = Hom(\Gamma, S^1) = (S^1)^2$.

The list $X = \{(2, 0), (0, 3), (-1, 1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given $T$ by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.

The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list $X$ is given in $\mathbb{R}^2$ by equations: $2x = 0, 3y = 0, -x + y = 0$.

If we replace $(0, 3)$ by $(0, 5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$. 
A list $X$ of elements of a finitely generated abelian group $\Gamma$ defines a \textit{(generalized) toric arrangement} $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $\text{Hom}(G, S^1)$, composed by a subgroup $H_e$ for every $e \in X$ (having codimension 0 if $e$ is torsion and 1 otherwise).

Example: $\Gamma = \mathbb{Z}^2$, $T = \text{Hom}(\Gamma, S^1) = (S^1)^2$.

The list $X = \{(2, 0), (0, 3), (-1, 1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given $T$ by the equations: $t^2 = 1$, $s^3 = 1$, $t^{-1}s = 1$.

The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list $X$ is given in $\mathbb{R}^2$ by equations: $2x = 0$, $3y = 0$, $-x + y = 0$.

If we replace $(0, 3)$ by $(0, 5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$. 
A list $X$ of elements of a finitely generated abelian group $\Gamma$ defines a (generalized) toric arrangement $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $\text{Hom}(G, S^1)$, composed by a subgroup $H_e$ for every $e \in X$ (having codimension 0 if $e$ is torsion and 1 otherwise).

Example: $\Gamma = \mathbb{Z}^2$, $T = \text{Hom}(\Gamma, S^1) = (S^1)^2$.

The list $X = \{(2, 0), (0, 3), (-1, 1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given $T$ by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.

The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list $X$ is given in $\mathbb{R}^2$ by equations: $2x = 0, 3y = 0, -x + y = 0$.

If we replace $(0, 3)$ by $(0, 5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$. 

Luca Moci (Marie Curie fellow of INdAM at Université de Paris 7)
A list $X$ of elements of a finitely generated abelian group $\Gamma$ defines a (generalized) toric arrangement $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $\text{Hom}(G, \mathbb{S}^1)$, composed by a subgroup $H_e$ for every $e \in X$ (having codimension 0 if $e$ is torsion and 1 otherwise).

Example: $\Gamma = \mathbb{Z}^2$, $T = \text{Hom}(\Gamma, \mathbb{S}^1) = (\mathbb{S}^1)^2$.
The list $X = \{(2, 0), (0, 3), (-1, 1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given $T$ by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.
The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list $X$ is given in $\mathbb{R}^2$ by equations: $2x = 0, 3y = 0, -x + y = 0$.

If we replace $(0, 3)$ by $(0, 5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$. 
Toric arrangements

A list $X$ of elements of a finitely generated abelian group $\Gamma$ defines a (generalized) toric arrangement $T_X$; this is a family of subgroups in the abelian compact Lie group $\text{Hom}(G, S^1)$, composed by a subgroup $H_e$ for every $e \in X$ (having codimension 0 if $e$ is torsion and 1 otherwise).

Example: $\Gamma = \mathbb{Z}^2$, $T = \text{Hom}(\Gamma, S^1) = (S^1)^2$.

The list $X = \{(2, 0), (0, 3), (-1, 1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $T_X$ given $T$ by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.

The hyperplane arrangement $H_X$ corresponding to the same list $X$ is given in $\mathbb{R}^2$ by equations: $2x = 0, 3y = 0, -x + y = 0$.

If we replace $(0, 3)$ by $(0, 5)$, we get the same $H_X$, but a different $T_X$. 

Luca Moci (Marie Curie fellow of INdAM at Université de Paris 7)
Matroids and arithmetic matroids

The characteristic polynomial $\chi_H$ of a hyperplane arrangement $H$ only depends on the "linear algebra" of the linear forms defining $H$. This linear algebra is captured by the combinatorial notion of matroid. To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_H$: $T(x, y) = \sum_{A \subseteq X} (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}$.

The characteristic polynomial $\chi_T$ of a toric arrangement $T$ depends not only on the linear algebra of the characters defining $T$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid. To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_T$: $M(x, y) = \sum_{A \subseteq X} m(A) (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}$ where in the realizable case $m(A) = |(\Gamma/\langle A \rangle)_{tors}|$. Has this applications to graph theory?
The characteristic polynomial $\chi_H$ of a hyperplane arrangement $H$ only depends on the "linear algebra" of the linear forms defining $H$. This linear algebra is captured by the combinatorial notion of matroid. To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_H$: $T(x, y) = \sum_{A \subseteq X} (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}$.

The characteristic polynomial $\chi_T$ of a toric arrangement $T$ depends not only on the linear algebra of the characters defining $T$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid. To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_T$: $M(x, y) = \sum_{A \subseteq X} m(A) (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}$ where in the realizable case $m(A) = |(\Gamma/\langle A \rangle)_{tor}|$.

Has this applications to graph theory?

Luca Moci (Marie Curie fellow of INdAM at Università di Padova)
The characteristic polynomial $\chi_H$ of a hyperplane arrangement $H$ only depends on the ”linear algebra” of the linear forms defining $H$. This linear algebra is captured by the combinatorial notion of matroid. To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_H$: 

$$T(x, y) = \sum_{A \subseteq X} (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}.$$

The characteristic polynomial $\chi_T$ of a toric arrangement $T$ depends not only on the linear algebra of the characters defining $T$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid. To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_T$: 

$$M(x, y) = \sum_{A \subseteq X} m(A) (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}$$

where in the realizable case $m(A) = |(\Gamma/\langle A \rangle)_{tors}|$. Has this applications to graph theory?
Matroids and arithmetic matroids

The characteristic polynomial $\chi_\mathcal{H}$ of a hyperplane arrangement $\mathcal{H}$ only depends on the ”linear algebra” of the linear forms defining $\mathcal{H}$. This linear algebra is captured by the combinatorial notion of matroid. To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_\mathcal{H}$: 

$$T(x, y) = \sum_{A \subseteq X} (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}.$$

The characteristic polynomial $\chi_\mathcal{T}$ of a toric arrangement $\mathcal{T}$ depends not only on the linear algebra of the characters defining $\mathcal{T}$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid.

To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_\mathcal{T}$: 

$$M(x, y) = \sum_{A \subseteq X} m(A) (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}$$

where in the realizable case $m(A) = |(\mathcal{F}/\langle A \rangle)_\text{tors}|$.

Has this applications to graph theory?
Matroids and arithmetic matroids

The characteristic polynomial $\chi_{\mathcal{H}}$ of a hyperplane arrangement $\mathcal{H}$ only depends on the "linear algebra" of the linear forms defining $\mathcal{H}$. This linear algebra is captured by the combinatorial notion of matroid. To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_{\mathcal{H}}$: $T(x, y) \doteq \sum_{A \subseteq X} (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}$.

The characteristic polynomial $\chi_{\mathcal{T}}$ of a toric arrangement $\mathcal{T}$ depends not only on the linear algebra of the characters defining $\mathcal{T}$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid. To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_{\mathcal{T}}$: $M(x, y) \doteq \sum_{A \subseteq X} m(A) (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}$ where in the realizable case $m(A) = |(\Gamma/\langle A \rangle)_{tors}|$.

Has this applications to graph theory?
Labelled graphs

Graph $G := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

For example, let $G$, where $G := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$, $R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the regular edges, $D := \{\{v_3, v_4\}\}$ the dotted edges; let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$, $\ell(\{v_3, v_4\}) = 6$ be the labels.
Labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

For example, let $\mathcal{G}$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$, $R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the regular edges, $D := \{\{v_3, v_4\}\}$ the dotted edges; let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$, $\ell(\{v_3, v_4\}) = 6$ be the labels.
Labelled graphs

Graph \( \mathcal{G} := (V, E) \) with a map \( \ell : E \mapsto \mathbb{Z}_{>0} \) and a partition \( E = R \sqcup D \).

For example, let \( \mathcal{G} \), where \( \mathcal{G} := (V, E), \ V := \{v_1, v_2, v_3, v_4\} \), \( R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\} \) the regular edges, \( D := \{\{v_3, v_4\}\} \) the dotted edges; let \( \ell(\{v_1, v_2\}) = 1, \ \ell(\{v_2, v_3\}) = 2, \ \ell(\{v_2, v_4\}) = 3, \ \ell(\{v_3, v_4\}) = 6 \) be the labels.
Labelled graphs

Graph \( \mathcal{G} := (V, E) \) with a map \( \ell : E \mapsto \mathbb{Z}_{>0} \) and a partition \( E = R \sqcup D \).

For example, let \( \mathcal{G} \), where \( \mathcal{G} := (V, E) \), \( V := \{v_1, v_2, v_3, v_4\} \), \( R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\} \) the regular edges, \( D := \{\{v_3, v_4\}\} \) the dotted edges; let \( \ell(\{v_1, v_2\}) = 1, \ell(\{v_2, v_3\}) = 2, \ell(\{v_2, v_4\}) = 3, \ell(\{v_3, v_4\}) = 6 \) be the labels.
Labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

For example, let $\mathcal{G}$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$, $R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the regular edges, $D := \{\{v_3, v_4\}\}$ the dotted edges; let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$, $\ell(\{v_3, v_4\}) = 6$ be the labels.
Oriented labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

For example, let $\mathcal{G}$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
$R := \{(v_1, v_2), (v_3, v_2), (v_2, v_4)\}$ the regular edges,
$D := \{(v_3, v_4)\}$ the dotted edges;
let $\ell((v_1, v_2)) = 1$, $\ell((v_3, v_2)) = 2$, $\ell((v_2, v_4)) = 3$,
$\ell((v_3, v_4)) = 6$ be the labels.
Deletion and contraction

Deletion of \( \{v_2, v_3\} \).

Contraction of \( \{v_2, v_3\} \).

What is a graphical toric arrangement?

June 12, 2012 9 / 14
Deletion and contraction

Deletion of \{v_2, v_3\}.

Contraction of \{v_2, v_3\}.
Arithmetic colorings

For our results we will consider only positive integers $q$ such that $L(G) \divides LCM \ell(e)$, $e \in E$ divides $q$. We will call such an integer admissible. A (proper) arithmetic $q$-coloring of a labelled graph $G$ is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

1. if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
2. if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The arithmetic chromatic polynomial $\chi_G(q)$ of $G$ is defined as the number of (proper) arithmetic $q$-colorings of $G$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.
Arithmetic colorings

For our results we will consider only positive integers $q$ such that $L(G) \div LCM \ell(e), \ e \in E$ divides $q$. We will call such an integer admissible. A (proper) arithmetic $q$-coloring of a labelled graph $G$ is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

1. if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
2. if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The arithmetic chromatic polynomial $\chi_G(q)$ of $G$ is defined as the number of (proper) arithmetic $q$-colorings of $G$. When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.
Arithmetic colorings

For our results we will consider only positive integers $q$ such that $L(G) \divides \operatorname{LCM} \ell(e)$, $e \in E$ divides $q$. We will call such an integer admissible. A (proper) arithmetic $q$-coloring of a labelled graph $G$ is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

1. if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
2. if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The arithmetic chromatic polynomial $\chi_G(q)$ of $G$ is defined as the number of (proper) arithmetic $q$-colorings of $G$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.
Arithmetic colorings

For our results we will consider only positive integers $q$ such that $L(G) \div LCM \ell(e)$, $e \in E$ divides $q$. We will call such an integer admissible.

A (proper) arithmetic $q$-coloring of a labelled graph $G$ is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

1. if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
2. if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The arithmetic chromatic polynomial $\chi_G(q)$ of $G$ is defined as the number of (proper) arithmetic $q$-colorings of $G$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.
For our results we will consider only positive integers $q$ such that $L(G) \equiv LCM \ell(e)$, $e \in E$ divides $q$. We will call such an integer admissible. A (proper) arithmetic $q$-coloring of a labelled graph $G$ is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

1. if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
2. if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The arithmetic chromatic polynomial $\chi_G(q)$ of $G$ is defined as the number of (proper) arithmetic $q$-colorings of $G$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial. We have:

$$2c(v_1) \neq 2c(v_2), \quad 3c(v_1) \neq 3c(v_2)$$

$$2c(v_2) = 2c(v_3).$$

We can color $v_1$ in $q$ ways, then $v_2$ in $q - 3 - 2 + 1$ ways, then $v_3$ in 2 ways, so $\chi_G(q) = 2q(q - 4) = 2q^2 - 8q$. 
Arithmetic colorings

For our results we will consider only positive integers \( q \) such that \( L(G) \mid LCM \ell(e) \), \( e \in E \) divides \( q \). We will call such an integer admissible.

A (proper) arithmetic \( q \)-coloring of a labelled graph \( G \) is a map \( c : V \to \mathbb{Z}/q\mathbb{Z} \) such that:

1. if \( e := \{u, v\} \in R \), then \( \ell(e) \cdot c(u) \neq \ell(e) \cdot c(v) \);
2. if \( e := \{u, v\} \in D \), then \( \ell(e) \cdot c(u) = \ell(e) \cdot c(v) \).

The arithmetic chromatic polynomial \( \chi_G(q) \) of \( G \) is defined as the number of (proper) arithmetic \( q \)-colorings of \( G \).

When \( D = \emptyset \) and \( \ell \equiv 1 \) we get the classical chromatic polynomial.

We have:

\[
2c(v_1) \neq 2c(v_2), \quad 3c(v_1) \neq 3c(v_2)
\]

\[
2c(v_2) = 2c(v_3).
\]

We can color \( v_1 \) in \( q \) ways, then \( v_2 \) in \( q - 3 - 2 + 1 \) ways, then \( v_3 \) in 2 ways, so \( \chi_G(q) = 2q(q - 4) = 2q^2 - 8q \).
Given an admissible $q$, a (nowhere zero) arithmetic $q$-flow on an oriented labelled graph $G_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

\[(1) \forall v \in V, \sum_{s(e) = v} \ell(e)f(e) = \sum_{t(e) = v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}\]

(2) for all $e \in R_\theta$, $f(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The arithmetic flow polynomial $\chi^*_G(q)$ of $G$ is defined as the number of (nowhere zero) arithmetic $q$-flows of $G_\theta$ (it doesn’t depend on $\theta$). When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.
Given an admissible $q$, a (nowhere zero) arithmetic $q$-flow on an oriented labelled graph $G_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

1. $\forall v \in V$, $\sum_{s(e) = v} \ell(e)f(e) = \sum_{t(e) = v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}$

2. for all $e \in R_\theta$, $f(e) \neq 0 \in \mathbb{Z}/q\mathbb{Z}$.

The arithmetic flow polynomial $\chi^*_G(q)$ of $G$ is defined as the number of (nowhere zero) arithmetic $q$-flows of $G_\theta$ (it doesn’t depend on $\theta$).

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.
Arithmetic flows

Given an admissible $q$, a (nowhere zero) arithmetic $q$-flow on an oriented labelled graph $G_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

\[(1) \forall v \in V, \sum_{s(e)=v} \ell(e)f(e) = \sum_{t(e)=v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}\]

\[(2) \text{for all } e \in R_\theta, f(e) \neq 0 \in \mathbb{Z}/q\mathbb{Z}.

The arithmetic flow polynomial $\chi^*_G(q)$ of $G$ is defined as the number of (nowhere zero) arithmetic $q$-flows of $G_\theta$ (it doesn’t depend on $\theta$).

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.
Given an admissible $q$, a (nowhere zero) arithmetic $q$-flow on an oriented labelled graph $G_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

1. $\forall v \in V, \sum_{s(e) = v} \ell(e)f(e) = \sum_{t(e) = v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}$

2. for all $e \in R_\theta$, $f(e) \neq 0 \in \mathbb{Z}/q\mathbb{Z}$.

The arithmetic flow polynomial $\chi^*(G)(q)$ of $G$ is defined as the number of (nowhere zero) arithmetic $q$-flows of $G_\theta$ (it doesn’t depend on $\theta$). When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.
Given an admissible \( q \), a (nowhere zero) **arithmetic \( q \)-flow** on an oriented labelled graph \( G_\theta \) is a map \( f : E_\theta \to (\mathbb{Z}/q\mathbb{Z}) \) such that:

\[
\forall v \in V, \quad \sum_{s(e)=v} \ell(e)f(e) = \sum_{t(e)=v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}
\]

(2) for all \( e \in R_\theta \), \( f(e) \neq 0 \in \mathbb{Z}/q\mathbb{Z} \).

The **arithmetic flow polynomial** \( \chi^*_G(q) \) of \( G \) is defined as the number of (nowhere zero) arithmetic \( q \)-flows of \( G_\theta \) (it doesn’t depend on \( \theta \)).

When \( D = \emptyset \) and \( \ell \equiv 1 \) we get the classical flow polynomial.
Given an admissible $q$, a (nowhere zero) arithmetic $q$-flow on an oriented labelled graph $G_\theta$ is a map $f : E_\theta \to (\mathbb{Z} / q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \quad \sum_{s(e) = v} \ell(e)f(e) = \sum_{t(e) = v} \ell(e)f(e) \in \mathbb{Z} / q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $f(e) \neq \bar{0} \in \mathbb{Z} / q\mathbb{Z}$.

The arithmetic flow polynomial $\chi^*_G(q)$ of $G$ is defined as the number of (nowhere zero) arithmetic $q$-flows of $G_\theta$ (it doesn’t depend on $\theta$). When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.
Arithmetic flows

Given an admissible \( q \), a (nowhere zero) arithmetic \( q \)-flow on an oriented labelled graph \( G_\theta \) is a map \( f : E_\theta \to (\mathbb{Z}/q\mathbb{Z}) \) such that:

\[
\forall v \in V, \quad \sum_{s(e) = v} \ell(e)f(e) = \sum_{t(e) = v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}
\]

(2) for all \( e \in R_\theta \), \( f(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z} \).

The arithmetic flow polynomial \( \chi^*_G(q) \) of \( G \) is defined as the number of (nowhere zero) arithmetic \( q \)-flows of \( G_\theta \) (it doesn’t depend on \( \theta \)).

When \( D = \emptyset \) and \( \ell \equiv 1 \) we get the classical flow polynomial.

The equation \( 2x - 3y = 0 \) has \( q \) solutions, but 4 of them are not nowhere zero.

We have that \( \chi^*_G(q) = 2(q - 4) = 2q - 8 \).

Luca Moci (Marie Curie fellow of INdAM at Universit`e de Paris 7)

What is a graphical toric arrangement?
Arithmetic flows

Given an admissible $q$, a (nowhere zero) arithmetic $q$-flow on an oriented labelled graph $G_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

\[
\forall v \in V, \sum_{s(e)=v} \ell(e)f(e) = \sum_{t(e)=v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}
\]

(2) for all $e \in R_\theta$, $f(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The arithmetic flow polynomial $\chi_G^*(q)$ of $G$ is defined as the number of (nowhere zero) arithmetic $q$-flows of $G_\theta$ (it doesn’t depend on $\theta$).

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

The equation $2x - 3y = 0$ has $q$ solutions, but 4 of them are not nowhere zero.

We have that $\chi_G^*(q) = 2(q - 4) = 2q - 8$. 
Graphical toric arrangements

We associate to $G$ a list of elements of a group in the following way. To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$

$x_e = (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots)$.

Then we look at the image of the list $X_R$ in the group $\Gamma := \mathbb{Z}^n/\langle X_D \rangle$.

This defines a graphical toric arrangement $T(G)$ in $\text{Hom}(\Gamma, S^1)$.

We will see that $\chi_{T(G)} = \chi_G$.

We consider the associated arithmetic Tutte polynomial

$M(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{rk(X) - rk(A)}(y - 1)^{|A| - rk(A)}$. 

What is a graphical toric arrangement?
We associate to $G$ a list of elements of a group in the following way. To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$
$x_e = (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots)$.

Then we look at the image of the list $X_R$ in the group $\Gamma := \mathbb{Z}^n/\langle X_D \rangle$. This defines a graphical toric arrangement $T(G)$ in $Hom(\Gamma, S^1)$.

We will see that $\chi_{T(G)} = \chi_G$.

We consider the associated arithmetic Tutte polynomial
$M(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{rk(X) - rk(A)}(y - 1)^{|A| - rk(A)}$. 

What is a graphical toric arrangement?
Graphical toric arrangements

We associate to $G$ a list of elements of a group in the following way. To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$

$$x_e = (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots).$$

Then we look at the image of the list $X_R$ in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$. This defines a graphical toric arrangement $T(G)$ in $\text{Hom}(\Gamma, S^1)$. We will see that $\chi_{T(G)} = \chi_G$.

We consider the associated arithmetic Tutte polynomial $M(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{rk(X) - rk(A)}(y - 1)^{|A| - rk(A)}$. 

Luca Moci (Marie Curie fellow of INdAM at Universitét de Paris 7)
Graphical toric arrangements

We associate to $G$ a list of elements of a group in the following way. To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$

$x_e = (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots).$

Then we look at the image of the list $X_R$ in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$. This defines a graphical toric arrangement $T(G)$ in $\text{Hom}(\Gamma, S^1)$.

We will see that $\chi_{T(G)} = \chi_G$.
We consider the associated arithmetic Tutte polynomial

$M(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{rk(X) - rk(A)}(y - 1)^{|A| - rk(A)}$. 

What is a graphical toric arrangement?
Graphical toric arrangements

We associate to $\mathcal{G}$ a list of elements of a group in the following way. To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$

$$x_e = (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots).$$

Then we look at the image of the list $X_R$ in the group $\Gamma := \mathbb{Z}^n/\langle X_D \rangle$. This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $\text{Hom}(\Gamma, S^1)$.

We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.

We consider the associated arithmetic Tutte polynomial

$$M(x, y) = \sum_{A \subseteq X} m(A) (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}.$$
Graphical toric arrangements

We associate to $G$ a list of elements of a group in the following way. To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$ $x_e = (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots)$. Then we look at the image of the list $X_R$ in the group $\Gamma := \mathbb{Z}^n/\langle X_D \rangle$. This defines a graphical toric arrangement $\mathcal{T}(G)$ in $Hom(\Gamma, S^1)$. We will see that $\chi_{\mathcal{T}(G)} = \chi_G$.

We consider the associated arithmetic Tutte polynomial $M(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{rk(X) - rk(A)}(y - 1)^{|A| - rk(A)}$.

We have $X_R = \{(1, -1, 0, 0), (0, -2, 2, 0), (0, 3, 0, -3)\} \subseteq \mathbb{Z}^4$ and $X_D = \{(0, 0, 6, -6)\} \subseteq \mathbb{Z}^4$, so $G := \mathbb{Z}^4/\langle (0, 0, 6, -6) \rangle$.

In this case $M_G(x, y) = 6x^2 + 18x + 6xy$. 

Luca Moci (Marie Curie fellow of INdAM at Université de Paris 7)
Graphical toric arrangements

We associate to $\mathcal{G}$ a list of elements of a group in the following way. To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$

$$x_e = (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots).$$

Then we look at the image of the list $X_R$ in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$. This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $\text{Hom}(\Gamma, S^1)$. We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.

We consider the associated arithmetic Tutte polynomial

$$M(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{rk(X) - rk(A)}(y - 1)^{|A| - rk(A)}.$$

We have $X_R = \{(1, -1, 0, 0), (0, -2, 2, 0), (0, 3, 0, -3)\} \subseteq \mathbb{Z}^4$ and $X_D = \{(0, 0, 6, -6)\} \subseteq \mathbb{Z}^4$, so $G := \mathbb{Z}^4 / \langle (0, 0, 6, -6) \rangle$.

In this case $M_G(x, y) = 6x^2 + 18x + 6xy$. 
We associate to $\mathcal{G}$ a list of elements of a group in the following way. To each edge $e = (v_i, v_j) \in E$ we associate the element of $\mathbb{Z}^n$

$$x_e = (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots).$$

Then we look at the image of the list $X_R$ in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$. This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $\text{Hom}(\Gamma, S^1)$.

We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.

We consider the associated arithmetic Tutte polynomial

$$M(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{rk(X) - rk(A)}(y - 1)^{|A| - rk(A)}.$$

We have $X_R = \{(1, -1, 0, 0), (0, -2, 2, 0), (0, 3, 0, -3)\} \subseteq \mathbb{Z}^4$ and $X_D = \{(0, 0, 6, -6)\} \subseteq \mathbb{Z}^4$, so $G := \mathbb{Z}^4 / \langle (0, 0, 6, -6) \rangle$.

In this case $M_G(x, y) = 6x^2 + 18x + 6xy$. 
Main results

Let $\bar{G} = (\bar{V}, \bar{E})$ be the graph obtained from $G = (V, E = R \cup D)$ by (classically) contracting the edges in $D$. Let $q$ be an admissible integer.

**Theorem (M.- D’Adderio)**

1. $\chi_G(q) = (-1)^{|\bar{V}| - k} q^k M_G(1 - q, 0)$.
2. $\chi_G^*(q) = (-1)^{|R| - |\bar{V}| + k} q^{|D| - |V| + |\bar{V}|} M_G(0, 1 - q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

**Theorem (Tutte)**

1. $\chi_G(q) = (-1)^{|V| - k} q^k T_G(1 - q, 0)$.
2. $\chi_G^*(q) = (-1)^{|E| - |V| + k} T_G(0, 1 - q)$.
Let $\overline{G} = (\overline{V}, \overline{E})$ be the graph obtained from $G = (V, E = R \cup D)$ by (classically) contracting the edges in $D$. Let $q$ be an admissible integer.

**Theorem (M.- D’Adderio)**

1. $\chi^*_G(q) = (-1)|\overline{V}|^{-k} q^k M_G(1 - q, 0)$.
2. $\chi^*_G(q) = (-1)|R|^{-|\overline{V}|+k} q^D - |V| + |\overline{V}| M_G(0, 1 - q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

**Theorem (Tutte)**

1. $\chi^*_G(q) = (-1)|V|^{-k} q^k T_G(1 - q, 0)$.
2. $\chi^*_G(q) = (-1)|E|^{-|V|+k} T_G(0, 1 - q)$.
Main results

Let $\bar{G} = (\bar{V}, \bar{E})$ be the graph obtained from $G = (V, E = R \cup D)$ by (classically) contracting the edges in $D$. Let $q$ be an admissible integer.

**Theorem (M.- D’Adderio)**

1. $\chi_G(q) = (-1)^{|\bar{V}|}q^k M_G(1 - q, 0)$.
2. $\chi_G^*(q) = (-1)^{|R| - |\bar{V}| + k} q^D - |V| + |\bar{V}| M_G(0, 1 - q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

**Theorem (Tutte)**

1. $\chi_G(q) = (-1)^{|V|}q^k T_G(1 - q, 0)$.
2. $\chi_G^*(q) = (-1)^{|E| - |V| + k} T_G(0, 1 - q)$.
Main results

Let $\bar{G} = (\bar{V}, \bar{E})$ be the graph obtained from $G = (V, E = R \cup D)$ by (classically) contracting the edges in $D$. Let $q$ be an admissible integer.

**Theorem (M.- D’Adderio)**

1. $\chi_G(q) = (-1)^{|V| - k} q^k M_G(1 - q, 0)$.
2. $\chi^*_G(q) = (-1)^{|R| - |V| + k} q^{|D| - |V| + |\bar{V}|} M_G(0, 1 - q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

**Theorem (Tutte)**

1. $\chi_G(q) = (-1)^{|V| - k} q^k T_G(1 - q, 0)$.
2. $\chi^*_G(q) = (-1)^{|E| - |V| + k} T_G(0, 1 - q)$.
Main results

Let \( \bar{G} = (\bar{V}, \bar{E}) \) be the graph obtained from \( G = (V, E = R \cup D) \) by (classically) contracting the edges in \( D \). Let \( q \) be an admissible integer.

**Theorem (M.- D’Adderio)**

1. \( \chi_G(q) = (-1)^{|\bar{V}|} q^k M_G(1 - q, 0) \).
2. \( \chi^*_G(q) = (-1)^{|R| - |\bar{V}| + k} q^{|D| - |V| + |\bar{V}|} M_G(0, 1 - q) \).

When \( D = \emptyset \) and \( \ell \equiv 1 \) we get the classical result:

**Theorem (Tutte)**

1. \( \chi_G(q) = (-1)^{|V|} q^k T_G(1 - q, 0) \).
2. \( \chi^*_G(q) = (-1)^{|E| - |V| + k} T_G(0, 1 - q) \).
Main results

Let $\overline{G} = (\overline{V}, \overline{E})$ be the graph obtained from $G = (V, E = R \cup D)$ by (classically) contracting the edges in $D$. Let $q$ be an admissible integer.

<table>
<thead>
<tr>
<th>Theorem (M.- D’Adderio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\chi_G(q) = (-1)^{</td>
</tr>
<tr>
<td>2. $\chi^*_G(q) = (-1)^{</td>
</tr>
</tbody>
</table>

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

<table>
<thead>
<tr>
<th>Theorem (Tutte)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\chi_G(q) = (-1)^{</td>
</tr>
<tr>
<td>2. $\chi^*_G(q) = (-1)^{</td>
</tr>
</tbody>
</table>
Main results

Let $\overline{G} = (\overline{V}, \overline{E})$ be the graph obtained from $G = (V, E = R \cup D)$ by (classically) contracting the edges in $D$. Let $q$ be an admissible integer.

**Theorem (M.- D’Adderio)**

1. $\chi_G(q) = (-1)^{|\overline{V}| - k} q^k M_G(1 - q, 0)$.
2. $\chi_G^*(q) = (-1)^{|R| - |\overline{V}| + k} q^{|D| - |V| + |\overline{V}|} M_G(0, 1 - q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

**Theorem (Tutte)**

1. $\chi_G(q) = (-1)^{|V| - k} q^k T_G(1 - q, 0)$.
2. $\chi_G^*(q) = (-1)^{|E| - |V| + k} T_G(0, 1 - q)$.

$$M_G(x, y) = 2x + 6 + 2y,$$ and therefore

$\chi_G(q) = 2q^2 - 8q$, $\chi_G^*(q) = 2q - 8$.
Main results

Let $\overline{G} = (\overline{V}, \overline{E})$ be the graph obtained from $G = (V, E = R \cup D)$ by (classically) contracting the edges in $D$. Let $q$ be an admissible integer.

**Theorem (M.- D’Adderio)**

1. $\chi_G(q) = (-1)^{|\overline{V}|} q^k M_G(1 - q, 0)$.
2. $\chi_G^*(q) = (-1)^{|R|} q^{|D| - |V| + |\overline{V}|} M_G(0, 1 - q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

**Theorem (Tutte)**

1. $\chi_G(q) = (-1)^{|V|} q^k T_G(1 - q, 0)$.
2. $\chi_G^*(q) = (-1)^{|E|} q^{|V| + k} T_G(0, 1 - q)$.

$M_G(x, y) = 2x + 6 + 2y$, and therefore $\chi_G(q) = 2q^2 - 8q, \chi_G^*(q) = 2q - 8$. 

Luca Moci (Marie Curie fellow of INdAM at the Université de Paris 7)
And if $q$ is not ammissible?

In general, the number of arithmetic colorings is a quasipolynomial in $q$. When $q \equiv 0 \mod L(G)$, it coincides with the arithmetic chromatic polynomial. When $q$ is coprime with $L(G)$, it coincides with the classical chromatic polynomial.

The same for the number of arithmetic flows.

Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and ”interpolates” between the classical and the arithmetic Tutte polynomial. This is defined for any list $X$ in a finitely generated abelian group $\Gamma$, in the more general language of ”multivariate Tutte polynomials”:

$$Q_X(q, v) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{|q \cdot \text{tors}(\frac{\Gamma}{\langle A \rangle})|} \prod_{e \in A} v_e.$$
Arithmetic quasipolynomials

And if \( q \) is not ammissible?
In general, the number of arithmetic colorings is a \textit{quasipolynomial} in \( q \).
When \( q \equiv 0 \mod L(\mathcal{G}) \), it coincides with the arithmetic chromatic polynomial.
When \( q \) is coprime with \( L(\mathcal{G}) \), it coincides with the classical chromatic polynomial.
The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an \textit{arithmetic Tutte quasipolynomial} that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.
This is defined for any list \( X \) in a finitely generated abelian group \( \Gamma \), in the more general language of "multivariate Tutte polynomials":

\[
Q_X(q, v) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{|q \cdot \text{tors}(\langle A \rangle)|} \prod_{e \in A} v_e.
\]

Luca Moci (Marie Curie fellow of INdAM at Université de Paris 7)

What is a graphical toric arrangement?
Arithmetic quasipolynomials

And if \( q \) is not ammissible?
In general, the number of arithmetic colorings is a quasipolynomial in \( q \). When \( q \equiv 0 \mod L(\mathcal{G}) \), it coincides with the arithmetic chromatic polynomial.
When \( q \) is coprime with \( L(\mathcal{G}) \), it coincides with the classical chromatic polynomial.
The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and ”interpolates” between the classical and the arithmetic Tutte polynomial.
This is defined for any list \( X \) in a finitely generated abelian group \( \Gamma \), in the more general language of ”multivariate Tutte polynomials”:

\[
Q_X(q,v) := q^{\text{rk}(\Gamma)} \sum_{A \subseteq X} q^{-\text{rk}(A)} \frac{m(A)}{|q \cdot \text{tors}(\frac{\Gamma}{\langle A \rangle})|} \prod_{e \in A} v_e.
\]
And if \( q \) is not ammissible?

In general, the number of arithmetic colorings is a quasipolynomial in \( q \).

When \( q \equiv 0 \mod L(G) \), it coincides with the arithmetic chromatic polynomial.

When \( q \) is coprime with \( L(G) \), it coincides with the classical chromatic polynomial.

The same for the number of arithmetic flows.

Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and ”interpolates” between the classical and the arithmetic Tutte polynomial.

This is defined for any list \( X \) in a finitely generated abelian group \( \Gamma \), in the more general language of ”multivariate Tutte polynomials”:

\[
Q_X(q, v) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{|q \cdot tors(\langle \Gamma \rangle)|} \prod_{e \in A} v_e.
\]
And if $q$ is not ammissible?
In general, the number of arithmetic colorings is a quasipolynomial in $q$.
When $q \equiv 0 \mod L(G)$, it coincides with the arithmetic chromatic polynomial.
When $q$ is coprime with $L(G)$, it coincides with the classical chromatic polynomial.
The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.
This is defined for any list $X$ in a finitely generated abelian group $\Gamma$, in the more general language of "multivariate Tutte polynomials":

$$Q_X(q, v) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{|q \cdot tors(\Gamma/\langle A \rangle)|} \prod_{e \in A} v_e.$$
Arithmetic quasipolynomials

And if $q$ is not admissible?
In general, the number of arithmetic colorings is a quasipolynomial in $q$. When $q \equiv 0 \mod L(G)$, it coincides with the arithmetic chromatic polynomial. When $q$ is coprime with $L(G)$, it coincides with the classical chromatic polynomial. The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and ”interpolates” between the classical and the arithmetic Tutte polynomial.

This is defined for any list $X$ in a finitely generated abelian group $\Gamma$, in the more general language of ”multivariate Tutte polynomials”:

\[ Q_X(q, v) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{|q \cdot \text{tors}(\langle A \rangle)|} \prod_{e \in A} v_e. \]
Arithmetic quasipolynomials

And if $q$ is not ammissible?
In general, the number of arithmetic colorings is a quasipolynomial in $q$.
When $q \equiv 0 \text{ mod } L(G)$, it coincides with the arithmetic chromatic polynomial.
When $q$ is coprime with $L(G)$, it coincides with the classical chromatic polynomial.
The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and ”interpolates” between the classical and the arithmetic Tutte polynomial.
This is defined for any list $X$ in a finitely generated abelian group $\Gamma$, in the more general language of ”multivariate Tutte polynomials”:

$$Q_X(q, v) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{|q \cdot tors\left(\frac{\Gamma}{\langle A \rangle}\right)|} \prod_{e \in A} v_e.$$
And if \( q \) is not admissible?
In general, the number of arithmetic colorings is a quasipolynomial in \( q \). When \( q \equiv 0 \bmod L(\mathcal{G}) \), it coincides with the arithmetic chromatic polynomial.
When \( q \) is coprime with \( L(\mathcal{G}) \), it coincides with the classical chromatic polynomial.
The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.
This is defined for any list \( X \) in a finitely generated abelian group \( \Gamma \), in the more general language of "multivariate Tutte polynomials":

\[
Q_X(q, v) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{|q \cdot tors(\frac{\Gamma}{\langle A \rangle})|} \prod_{e \in A} v_e.
\]