What is a graphical toric arrangement?

Arrangements in Pyrénées

Luca Moci

Marie Curie fellow of INdAM at Université de Paris 7

June 12, 2012

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j)\, \forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x, y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j)\, \forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V,\ \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x,y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j)\, \forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x,y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j)\, \forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V,\ \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x,y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j)\, \forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x,y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j)\,\forall(i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x,y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \, \forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G \setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x, y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \, \forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G \setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x, y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \, \forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G \setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \ \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x, y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j) \, \forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x, y)$.

Chromatic polynomial and flow polynomial

Let G be a graph, V the set of vertices, E the set of edges.
A $q-$coloring is a map $c : V \to \mathbb{Z}/q$. It is proper if $c(i) \neq c(j)\,\forall (i,j) \in E$.
FACT: the number $\chi_G(q)$ of proper $q-$colorings of G is a polynomial in q, which we call the chromatic polynomial of G.
FACT (deletion-contraction): $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$.

Given an orientation of G, a $q-$flow is a map $f : E \to \mathbb{Z}/q$ such that

$$\text{for every } v \in V, \sum_{s(e)=v} f(e) = \sum_{t(e)=v} f(e).$$

It is nowhere zero if $f(e) \neq 0 \forall e \in E$.
FACT: the number $\chi^*_G(q)$ of nowhere zero $q-$flows of G is a polynomial in q, which we call the flow polynomial of G.
Again, we have deletion-contraction.
What is most general deletion-contraction invariant of a graph?
The Tutte polynomial $T_G(x, y)$.

The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$.
Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Moebius funtion μ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \widehat{0}) q^{dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).

The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$.
Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Moebius funtion μ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \widehat{0}) q^{dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).

The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$.
Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Moebius funtion μ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \widehat{0}) q^{dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).

The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$.
Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Moebius funtion μ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \widehat{0}) q^{dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).

The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$.
Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Moebius funtion μ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \widehat{0}) q^{dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).

The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$.
Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Moebius funtion μ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \widehat{0}) q^{dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).

The characteristic polynomial

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{K}^n$.
Its intersection lattice $L(\mathcal{H})$ is a ranked poset, hence it has a Moebius funtion μ and a characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{L \in L(\mathcal{H})} \mu(L, \widehat{0}) q^{dim(L)}.$$

The polynomial $\chi_{\mathcal{H}}(q)$ contains much information on the complement of the arrangement $\mathcal{H}$, such as:

- the number of chambers and bounded chambers (if $\mathbb{K} = \mathbb{R}$);
- the Betti numbers of the complement (if $\mathbb{K} = \mathbb{C}$);
- the number of points in the complement (if $\mathbb{K} = \mathbb{F}_q$).

Graphical hyperplane arrangement

Let $G = (V, E)$ be a graph and $n = |V|$.
To G we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.
FACT: $\chi_{\mathcal{H}(G)} = \chi_G$.
We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$.
Moreover, if G is planar, it exists G^* such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.
So the question arises naturally:

Problem

What is a graphical toric arrangement?

Graphical hyperplane arrangement

Let $G = (V, E)$ be a graph and $n = |V|$.
To G we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.
FACT: $\chi_{\mathcal{H}(G)} = \chi_G$.
We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$.
Moreover, if G is planar, it exists G^* such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.
So the question arises naturally:

Problem

What is a graphical toric arrangement?

Graphical hyperplane arrangement

Let $G = (V, E)$ be a graph and $n = |V|$.
To G we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.
FACT: $\chi_{\mathcal{H}(G)} = \chi_G$.
We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi_G^*$.
Moreover, if G is planar, it exists G^* such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.
So the question arises naturally:

Problem

What is a graphical toric arrangement?

Graphical hyperplane arrangement

Let $G = (V, E)$ be a graph and $n = |V|$.
To G we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.
FACT: $\chi_{\mathcal{H}(G)} = \chi_G$.
We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$.
Moreover, if G is planar, it exists G^* such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.
So the question arises naturally:

Problem

What is a graphical toric arrangement?

Graphical hyperplane arrangement

Let $G = (V, E)$ be a graph and $n = |V|$.
To G we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.
FACT: $\chi_{\mathcal{H}(G)} = \chi_G$.
We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi_G^*$.
Moreover, if G is planar, it exists G^* such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.
So the question arises naturally:

Problem

What is a graphical toric arrangement?

Graphical hyperplane arrangement

Let $G = (V, E)$ be a graph and $n = |V|$.
To G we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.
FACT: $\chi_{\mathcal{H}(G)} = \chi_G$.
We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$.
Moreover, if G is planar, it exists G^* such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.
So the question arises naturally:

Problem

What is a graphical toric arrangement?

Graphical hyperplane arrangement

Let $G = (V, E)$ be a graph and $n = |V|$.
To G we associate a graphical hyperplane arrangement $\mathcal{H}(G)$ by taking, for every $(i, j) \in E$, the hyperplane of equation $e_i = e_j$ in $\mathbb{K}^n$.
FACT: $\chi_{\mathcal{H}(G)} = \chi_G$.
We can also define a dual arrangement $\mathcal{H}^*(G)$ such that $\chi_{\mathcal{H}^*(G)} = \chi^*_G$.
Moreover, if G is planar, it exists G^* such that $\mathcal{H}^*(G) = \mathcal{H}(G^*)$.

In recent years, toric arrangements have been studied intensively.
So the question arises naturally:

Problem

What is a graphical toric arrangement?

Toric arrangements

A list X of elements of a finitely generated abelian group Γ defines a (generalized) toric arrangement $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $Hom(G, \mathbb{S}^1)$, composed by a subgroup H_e for every $e \in X$ (having codimension 0 if e is torsion and 1 otherwise).

Example: $\Gamma = \mathbb{Z}^2$, $T = Hom(\Gamma, \mathbb{S}^1) = (\mathbb{S}^1)^2$.

The list $X = \{(2,0), (0,3), (-1,1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given T by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.

The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list X is given in $\mathbb{R}^2$ by equations: $2x = 0$, $3y = 0$, $-x + y = 0$.

If we replace $(0,3)$ by $(0,5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$.

Toric arrangements

A list X of elements of a finitely generated abelian group Γ defines a (generalized) toric arrangement $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $Hom(G, \mathbb{S}^1)$, composed by a subgroup H_e for every $e \in X$ (having codimension 0 if e is torsion and 1 otherwise).
Example: $\Gamma = \mathbb{Z}^2$, $T = Hom(\Gamma, \mathbb{S}^1) = (\mathbb{S}^1)^2$.
The list $X = \{(2,0),(0,3),(-1,1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given T by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.
The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list X is given in $\mathbb{R}^2$ by equations: $2x = 0$, $3y = 0$, $-x + y = 0$.

If we replace $(0,3)$ by $(0,5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$.

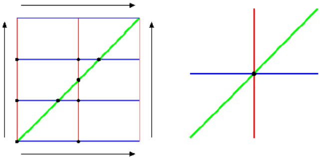
Toric arrangements

A list X of elements of a finitely generated abelian group Γ defines a (generalized) toric arrangement $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $Hom(G, \mathbb{S}^1)$, composed by a subgroup H_e for every $e \in X$ (having codimension 0 if e is torsion and 1 otherwise).
Example: $\Gamma = \mathbb{Z}^2$, $T = Hom(\Gamma, \mathbb{S}^1) = (\mathbb{S}^1)^2$.
The list $X = \{(2,0), (0,3), (-1,1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given T by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.
The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list X is given in $\mathbb{R}^2$ by equations: $2x = 0$, $3y = 0$, $-x + y = 0$.

If we replace $(0,3)$ by $(0,5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$.

Toric arrangements

A list X of elements of a finitely generated abelian group Γ defines a (generalized) toric arrangement $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $Hom(G, \mathbb{S}^1)$, composed by a subgroup H_e for every $e \in X$ (having codimension 0 if e is torsion and 1 otherwise).
Example: $\Gamma = \mathbb{Z}^2$, $T = Hom(\Gamma, \mathbb{S}^1) = (\mathbb{S}^1)^2$.
The list $X = \{(2,0),(0,3),(-1,1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given T by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.
The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list X is given in $\mathbb{R}^2$ by equations: $2x = 0$, $3y = 0$, $-x + y = 0$.

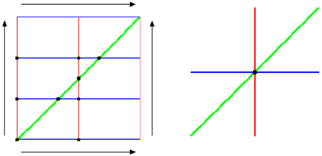


toric arrangement hyperplane arrangement

If we replace $(0,3)$ by $(0,5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$.

Toric arrangements

A list X of elements of a finitely generated abelian group Γ defines a (generalized) toric arrangement $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $Hom(G, \mathbb{S}^1)$, composed by a subgroup H_e for every $e \in X$ (having codimension 0 if e is torsion and 1 otherwise).
Example: $\Gamma = \mathbb{Z}^2$, $T = Hom(\Gamma, \mathbb{S}^1) = (\mathbb{S}^1)^2$.
The list $X = \{(2,0), (0,3), (-1,1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given T by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.
The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list X is given in $\mathbb{R}^2$ by equations: $2x = 0$, $3y = 0$, $-x + y = 0$.

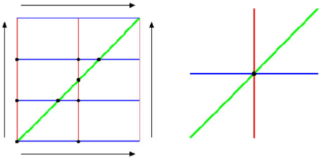


toric arrangement hyperplane arrangement

If we replace $(0,3)$ by $(0,5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$.

Toric arrangements

A list X of elements of a finitely generated abelian group Γ defines a (generalized) toric arrangement $\mathcal{T}_X$; this is a family of subgroups in the abelian compact Lie group $Hom(G, \mathbb{S}^1)$, composed by a subgroup H_e for every $e \in X$ (having codimension 0 if e is torsion and 1 otherwise).
Example: $\Gamma = \mathbb{Z}^2$, $T = Hom(\Gamma, \mathbb{S}^1) = (\mathbb{S}^1)^2$.
The list $X = \{(2,0), (0,3), (-1,1)\} \subset \mathbb{Z}^2$ defines the toric arrangement $\mathcal{T}_X$ given T by the equations: $t^2 = 1, s^3 = 1, t^{-1}s = 1$.
The hyperplane arrangement $\mathcal{H}_X$ corresponding to the same list X is given in $\mathbb{R}^2$ by equations: $2x = 0$, $3y = 0$, $-x + y = 0$.



toric arrangement hyperplane arrangement

If we replace $(0,3)$ by $(0,5)$, we get the same $\mathcal{H}_X$, but a different $\mathcal{T}_X$.

Matroids and arithmetic matroids

The characteristic polynomial $\chi_{\mathcal{H}}$ of a hyperplane arrangement $\mathcal{H}$ only depends on the "linear algebra" of the linear forms defining $\mathcal{H}$.
This linear algebra is captured by the combinatorial notion of matroid.
To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_{\mathcal{H}}$: $T(x,y) \doteq \sum_{A \subseteq X} (x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

The characteristic polynomial $\chi_{\mathcal{T}}$ of a toric arrangement $\mathcal{T}$ depends not only on the linear algebra of the characters defining $\mathcal{T}$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid.
To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_{\mathcal{T}}$:
$M(x,y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$
where in the realizable case $m(A) = |(\Gamma/\langle A \rangle)_{tors}|$.
Has this applications to graph theory?

Matroids and arithmetic matroids

The characteristic polynomial $\chi_{\mathcal{H}}$ of a hyperplane arrangement $\mathcal{H}$ only depends on the "linear algebra" of the linear forms defining $\mathcal{H}$.
This linear algebra is captured by the combinatorial notion of matroid.
To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_{\mathcal{H}}$: $T(x,y) \doteq \sum_{A \subseteq X} (x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

The characteristic polynomial $\chi_{\mathcal{T}}$ of a toric arrangement $\mathcal{T}$ depends not only on the linear algebra of the characters defining $\mathcal{T}$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid.
To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_{\mathcal{T}}$:
$M(x,y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$
where in the realizable case $m(A) = |(\Gamma / \langle A \rangle)_{tors}|$.
Has this applications to graph theory?

Matroids and arithmetic matroids

The characteristic polynomial $\chi_{\mathcal{H}}$ of a hyperplane arrangement $\mathcal{H}$ only depends on the "linear algebra" of the linear forms defining $\mathcal{H}$.
This linear algebra is captured by the combinatorial notion of matroid.
To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_{\mathcal{H}}$: $T(x,y) \doteq \sum_{A\subseteq X}(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

The characteristic polynomial $\chi_{\mathcal{T}}$ of a toric arrangement $\mathcal{T}$ depends not only on the linear algebra of the characters defining $\mathcal{T}$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid.
To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_{\mathcal{T}}$:
$M(x,y) \doteq \sum_{A\subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$
where in the realizable case $m(A) = |(\Gamma/\langle A\rangle)_{tors}|$.
Has this applications to graph theory?

Matroids and arithmetic matroids

The characteristic polynomial $\chi_{\mathcal{H}}$ of a hyperplane arrangement $\mathcal{H}$ only depends on the "linear algebra" of the linear forms defining $\mathcal{H}$.
This linear algebra is captured by the combinatorial notion of matroid.
To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_{\mathcal{H}}$: $T(x, y) \doteq \sum_{A \subseteq X} (x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

The characteristic polynomial $\chi_{\mathcal{T}}$ of a toric arrangement $\mathcal{T}$ depends not only on the linear algebra of the characters defining $\mathcal{T}$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid.
To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_{\mathcal{T}}$:
$M(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$
where in the realizable case $m(A) = |(\Gamma / \langle A \rangle)_{tors}|$.
Has this applications to graph theory?

Matroids and arithmetic matroids

The characteristic polynomial $\chi_{\mathcal{H}}$ of a hyperplane arrangement $\mathcal{H}$ only depends on the "linear algebra" of the linear forms defining $\mathcal{H}$.
This linear algebra is captured by the combinatorial notion of matroid.
To every matroid one associates a Tutte polynomial, which can be specialized to $\chi_{\mathcal{H}}$: $T(x,y) \doteq \sum_{A \subseteq X} (x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

The characteristic polynomial $\chi_{\mathcal{T}}$ of a toric arrangement $\mathcal{T}$ depends not only on the linear algebra of the characters defining $\mathcal{T}$, but also on their arithmetics. We tried to capture the linear algebra AND the arithmetics by the notion of arithmetic matroid.
To every arithmetic matroid we associated an arithmetic Tutte polynomial, which can be specialized to $\chi_{\mathcal{T}}$:
$M(x,y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$
where in the realizable case $m(A) = |(\Gamma/\langle A \rangle)_{tors}|$.
Has this applications to graph theory?

Labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

For example, let $\mathcal{G}$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
$R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the *regular edges*,
$D := \{\{v_3, v_4\}\}$ the *dotted edges*;
let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$,
$\ell(\{v_3, v_4\}) = 6$ be the *labels*.

Labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

For example, let $\mathcal{G}$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
$R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the *regular edges*,
$D := \{\{v_3, v_4\}\}$ the *dotted edges*;
let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$,
$\ell(\{v_3, v_4\}) = 6$ be the *labels*.

Labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

For example, let $\mathcal{G}$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
$R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the *regular edges*,
$D := \{\{v_3, v_4\}\}$ the *dotted edges*;
let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$,
$\ell(\{v_3, v_4\}) = 6$ be the *labels*.

Labelled graphs

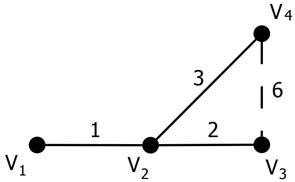
Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

For example, let $\mathcal{G}$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
$R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the *regular edges*,
$D := \{\{v_3, v_4\}\}$ the *dotted edges*;
let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$,
$\ell(\{v_3, v_4\}) = 6$ be the *labels*.

Labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

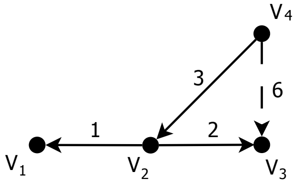
For example, let $\mathcal{G}$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
$R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the *regular edges*,
$D := \{\{v_3, v_4\}\}$ the *dotted edges*;
let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$,
$\ell(\{v_3, v_4\}) = 6$ be the *labels*.



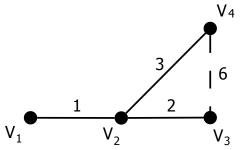
Oriented labelled graphs

Graph $\mathcal{G} := (V, E)$ with a map $\ell : E \mapsto \mathbb{Z}_{>0}$ and a partition $E = R \sqcup D$.

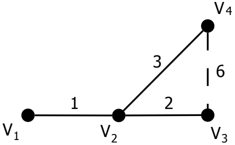
For example, let $\mathcal{G}$, where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
$R := \{(v_1, v_2), (v_3, v_2), (v_2, v_4)\}$ the *regular edges*,
$D := \{(v_3, v_4)\}$ the *dotted edges*;
let $\ell((v_1, v_2)) = 1$, $\ell((v_3, v_2)) = 2$, $\ell((v_2, v_4)) = 3$,
$\ell((v_3, v_4)) = 6$ be the *labels*.

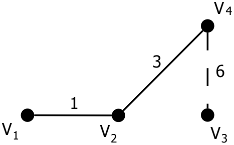


Deletion and contraction

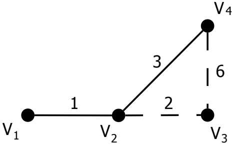


Deletion and contraction





Deletion of $\{v_2, v_3\}$.



Contraction of $\{v_2, v_3\}$.

Arithmetic colorings

For our results we will consider only positive integers q such that $L(\mathcal{G}) \doteq LCM\, \ell(e),\ e \in E$ divides q. We will call such an integer admissible.

A (proper) arithmetic q-coloring of a labelled graph $\mathcal{G}$ is a map $c : V \to \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The arithmetic chromatic polynomial $\chi_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (proper) arithmetic q-colorings of $\mathcal{G}$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.

Arithmetic colorings

For our results we will consider only positive integers q such that $L(\mathcal{G}) \doteq LCM\,\ell(e),\ e \in E$ divides q. We will call such an integer admissible.
A (proper) arithmetic q-coloring of a labelled graph $\mathcal{G}$ is a map $c : V \to \mathbb{Z}/q\mathbb{Z}$ such that:
(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.
The arithmetic chromatic polynomial $\chi_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (proper) arithmetic q-colorings of $\mathcal{G}$.
When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.

Arithmetic colorings

For our results we will consider only positive integers q such that $L(\mathcal{G}) \doteq LCM\, \ell(e),\ e \in E$ divides q. We will call such an integer admissible.
A (proper) arithmetic q-coloring of a labelled graph $\mathcal{G}$ is a map $c : V \to \mathbb{Z}/q\mathbb{Z}$ such that:
(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.
The arithmetic chromatic polynomial $\chi_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (proper) arithmetic q-colorings of $\mathcal{G}$.
When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.

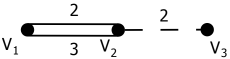
Arithmetic colorings

For our results we will consider only positive integers q such that $L(\mathcal{G}) \doteq LCM\,\ell(e),\ e \in E$ divides q. We will call such an integer admissible.
A (proper) arithmetic q-coloring of a labelled graph $\mathcal{G}$ is a map $c : V \to \mathbb{Z}/q\mathbb{Z}$ such that:
(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.
The arithmetic chromatic polynomial $\chi_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (proper) arithmetic q-colorings of $\mathcal{G}$.
When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.

Arithmetic colorings

For our results we will consider only positive integers q such that $L(\mathcal{G}) \doteq LCM\, \ell(e),\ e \in E$ divides q. We will call such an integer admissible.
A (proper) arithmetic q-coloring of a labelled graph $\mathcal{G}$ is a map $c : V \to \mathbb{Z}/q\mathbb{Z}$ such that:
(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.
The arithmetic chromatic polynomial $\chi_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (proper) arithmetic q-colorings of $\mathcal{G}$.
When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.
We have:

$$2c(v_1) \neq 2c(v_2)\ ,\ 3c(v_1) \neq 3c(v_2)$$



$$2c(v_2) = 2c(v_3).$$

We can color v_1 in q ways, then v_2 in $q - 3 - 2 + 1$ ways, then v_3 in 2 ways, so $\chi_{\mathcal{G}}(q) = 2q(q-4) = 2q^2 - 8q$.

Arithmetic colorings

For our results we will consider only positive integers q such that $L(\mathcal{G}) \doteq LCM\, \ell(e),\ e \in E$ divides q. We will call such an integer admissible.

A (proper) arithmetic q-coloring of a labelled graph $\mathcal{G}$ is a map $c : V \to \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

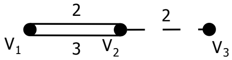
(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The arithmetic chromatic polynomial $\chi_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (proper) arithmetic q-colorings of $\mathcal{G}$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical chromatic polynomial.

We have:

$$2c(v_1) \neq 2c(v_2)\ ,\ 3c(v_1) \neq 3c(v_2)$$



$$2c(v_2) = 2c(v_3).$$

We can color v_1 in q ways, then v_2 in $q - 3 - 2 + 1$ ways, then v_3 in 2 ways, so $\chi_{\mathcal{G}}(q) = 2q(q-4) = 2q^2 - 8q$.

Arithmetic flows

Given an admissible q, a (nowhere zero) arithmetic q-flow on an oriented labelled graph $\mathcal{G}_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1)\forall v \in V, \sum_{s(e)=v} \ell(e)f(e) = \sum_{t(e)=v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $f(e) \neq \overline{0} \in \mathbb{Z}/q\mathbb{Z}$.
The *arithmetic flow polynomial* $\chi^*_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (nowhere zero) arithmetic q-flows of $\mathcal{G}_\theta$ (it doesn't depend on θ).
When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

Arithmetic flows

Given an admissible q, a (nowhere zero) arithmetic q-flow on an oriented labelled graph $\mathcal{G}_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{s(e)=v} \ell(e) f(e) = \sum_{t(e)=v} \ell(e) f(e) \in \mathbb{Z}/q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $f(e) \neq \overline{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi^*_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (nowhere zero) arithmetic q-flows of $\mathcal{G}_\theta$ (it doesn't depend on θ).

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

Given an admissible q, a (nowhere zero) arithmetic q-flow on an oriented labelled graph $\mathcal{G}_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1)\forall v \in V, \sum_{s(e)=v} \ell(e)f(e) = \sum_{t(e)=v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $f(e) \neq \overline{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi^*_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (nowhere zero) arithmetic q-flows of $\mathcal{G}_\theta$ (it doesn't depend on θ).

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

Arithmetic flows

Given an admissible q, a (nowhere zero) arithmetic q-flow on an oriented labelled graph $\mathcal{G}_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1)\forall v \in V, \sum_{s(e)=v} \ell(e)f(e) = \sum_{t(e)=v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $f(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi^*_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (nowhere zero) arithmetic q-flows of $\mathcal{G}_\theta$ (it doesn't depend on θ).

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

Arithmetic flows

Given an admissible q, a (nowhere zero) arithmetic q-flow on an oriented labelled graph $\mathcal{G}_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1)\forall v \in V, \sum_{s(e)=v} \ell(e)f(e) = \sum_{t(e)=v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $f(e) \neq \overline{0} \in \mathbb{Z}/q\mathbb{Z}$.
The *arithmetic flow polynomial* $\chi^*_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (nowhere zero) arithmetic q-flows of $\mathcal{G}_\theta$ (it doesn't depend on θ).
When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

Arithmetic flows

Given an admissible q, a (nowhere zero) arithmetic q-flow on an oriented labelled graph $\mathcal{G}_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{s(e)=v} \ell(e) f(e) = \sum_{t(e)=v} \ell(e) f(e) \in \mathbb{Z}/q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $f(e) \neq \overline{0} \in \mathbb{Z}/q\mathbb{Z}$.
The *arithmetic flow polynomial* $\chi^*_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (nowhere zero) arithmetic q-flows of $\mathcal{G}_\theta$ (it doesn't depend on θ).
When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

Arithmetic flows

Given an admissible q, a (nowhere zero) arithmetic q-flow on an oriented labelled graph $\mathcal{G}_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

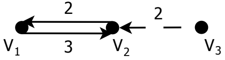
$$(1)\forall v \in V, \sum_{s(e)=v} \ell(e)f(e) = \sum_{t(e)=v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $f(e) \neq \overline{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi^*_\mathcal{G}(q)$ of $\mathcal{G}$ is defined as the number of (nowhere zero) arithmetic q-flows of $\mathcal{G}_\theta$ (it doesn't depend on θ).

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

The equation $2x - 3y = 0$ has q solutions, but 4 of them are not nowhere zero.



We have that $\chi^*_\mathcal{G}(q) = 2(q-4) = 2q-8$.

Arithmetic flows

Given an admissible q, a (nowhere zero) arithmetic q-flow on an oriented labelled graph $\mathcal{G}_\theta$ is a map $f : E_\theta \to (\mathbb{Z}/q\mathbb{Z})$ such that:

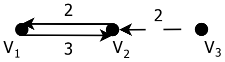
$$(1)\forall v \in V, \sum_{s(e)=v} \ell(e)f(e) = \sum_{t(e)=v} \ell(e)f(e) \in \mathbb{Z}/q\mathbb{Z}$$

(2) for all $e \in R_\theta$, $f(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi^*_{\mathcal{G}}(q)$ of $\mathcal{G}$ is defined as the number of (nowhere zero) arithmetic q-flows of $\mathcal{G}_\theta$ (it doesn't depend on θ).

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical flow polynomial.

The equation $2x - 3y = 0$ has q solutions, but 4 of them are not nowhere zero.



We have that $\chi^*_{\mathcal{G}}(q) = 2(q-4) = 2q - 8$.

Graphical toric arrangements

We associate to $\mathcal{G}$ a list of elements of a group in the following way.

To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$ $x_e \doteq (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots)$.

Then we look at the image of the list X_R in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$.

This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $Hom(\Gamma, S^1)$.

We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.

We consider the associated arithmetic Tutte polynomial $M(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

Graphical toric arrangements

We associate to $\mathcal{G}$ a list of elements of a group in the following way.
To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$
$x_e \doteq (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots)$.
Then we look at the image of the list X_R in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$.
This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $Hom(\Gamma, S^1)$.
We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.
We consider the associated arithmetic Tutte polynomial
$M(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

Graphical toric arrangements

We associate to $\mathcal{G}$ a list of elements of a group in the following way.
To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$
$x_e \doteq (0, \dots, 0, \ell(e), 0, \dots, 0, -\ell(e), 0, \dots)$.
Then we look at the image of the list X_R in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$.
This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $Hom(\Gamma, S^1)$.
We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.
We consider the associated arithmetic Tutte polynomial
$M(x,y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

Graphical toric arrangements

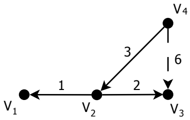
We associate to $\mathcal{G}$ a list of elements of a group in the following way.
To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$
$x_e \doteq (0, \dots, 0, \ell(e), 0, \dots, 0, -\ell(e), 0, \dots)$.
Then we look at the image of the list X_R in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$.
This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $Hom(\Gamma, S^1)$.
We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.
We consider the associated arithmetic Tutte polynomial
$M(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

Graphical toric arrangements

We associate to $\mathcal{G}$ a list of elements of a group in the following way.
To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$
$x_e \doteq (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots)$.
Then we look at the image of the list X_R in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$.
This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $Hom(\Gamma, S^1)$.
We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.
We consider the associated arithmetic Tutte polynomial
$M(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.

Graphical toric arrangements

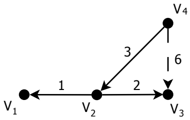
We associate to $\mathcal{G}$ a list of elements of a group in the following way.
To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$
$x_e \doteq (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots)$.
Then we look at the image of the list X_R in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$.
This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $Hom(\Gamma, S^1)$.
We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.
We consider the associated arithmetic Tutte polynomial
$M(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.



We have $X_R = \{(1, -1, 0, 0), (0, -2, 2, 0), (0, 3, 0, -3)\} \subseteq \mathbb{Z}^4$ and
$X_D = \{(0, 0, 6, -6)\} \subseteq \mathbb{Z}^4$, so $G := \mathbb{Z}^4 / \langle (0, 0, 6, -6) \rangle$.
In this case $M_{\mathcal{G}}(x, y) = 6x^2 + 18x + 6xy$.

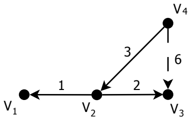
Graphical toric arrangements

We associate to $\mathcal{G}$ a list of elements of a group in the following way.
To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$
$x_e \doteq (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots)$.
Then we look at the image of the list X_R in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$.
This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $Hom(\Gamma, S^1)$.
We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.
We consider the associated arithmetic Tutte polynomial
$M(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.



We have $X_R = \{(1, -1, 0, 0), (0, -2, 2, 0), (0, 3, 0, -3)\} \subseteq \mathbb{Z}^4$ and
$X_D = \{(0, 0, 6, -6)\} \subseteq \mathbb{Z}^4$, so $G := \mathbb{Z}^4 / \langle (0, 0, 6, -6) \rangle$.
In this case $M_{\mathcal{G}}(x, y) = 6x^2 + 18x + 6xy$.

We associate to $\mathcal{G}$ a list of elements of a group in the following way.
To each edge $e = (v_i, v_j) \in E_\theta$ we associate the element of $\mathbb{Z}^n$
$x_e \doteq (0, \ldots, 0, \ell(e), 0, \ldots, 0, -\ell(e), 0, \ldots)$.
Then we look at the image of the list X_R in the group $\Gamma := \mathbb{Z}^n / \langle X_D \rangle$.
This defines a graphical toric arrangement $\mathcal{T}(\mathcal{G})$ in $Hom(\Gamma, S^1)$.
We will see that $\chi_{\mathcal{T}(\mathcal{G})} = \chi_{\mathcal{G}}$.
We consider the associated arithmetic Tutte polynomial
$M(x, y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}$.



We have $X_R = \{(1, -1, 0, 0), (0, -2, 2, 0), (0, 3, 0, -3)\} \subseteq \mathbb{Z}^4$ and
$X_D = \{(0, 0, 6, -6)\} \subseteq \mathbb{Z}^4$, so $G := \mathbb{Z}^4 / \langle (0, 0, 6, -6) \rangle$.
In this case $M_{\mathcal{G}}(x, y) = 6x^2 + 18x + 6xy$.

Main results

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$ be the graph obtained from $\mathcal{G} = (V, E = R \cup D)$ by (classically) contracting the edges in D. Let q be an admissible integer.

Theorem (M.- D'Adderio)

1. $\chi_{\mathcal{G}}(q) = (-1)^{|\overline{V}|-k} q^k M_{\mathcal{G}}(1-q, 0)$.
2. $\chi^*_{\mathcal{G}}(q) = (-1)^{|R|-|\overline{V}|+k} q^{|D|-|V|+|\overline{V}|} M_{\mathcal{G}}(0, 1-q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

Theorem (Tutte)

1. $\chi_G(q) = (-1)^{|V|-k} q^k T_G(1-q, 0)$.
2. $\chi^*_G(q) = (-1)^{|E|-|V|+k} T_G(0, 1-q)$.

Main results

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$ be the graph obtained from $\mathcal{G} = (V, E = R \cup D)$ by (classically) contracting the edges in D. Let q be an admissible integer.

Theorem (M.- D'Adderio)

1. $\chi_{\mathcal{G}}(q) = (-1)^{|\overline{V}|-k} q^k M_{\mathcal{G}}(1-q, 0)$.
2. $\chi^*_{\mathcal{G}}(q) = (-1)^{|R|-|\overline{V}|+k} q^{|D|-|V|+|\overline{V}|} M_{\mathcal{G}}(0, 1-q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

Theorem (Tutte)

1. $\chi_G(q) = (-1)^{|V|-k} q^k T_G(1-q, 0)$.
2. $\chi^*_G(q) = (-1)^{|E|-|V|+k} T_G(0, 1-q)$.

Main results

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$ be the graph obtained from $\mathcal{G} = (V, E = R \cup D)$ by (classically) contracting the edges in D. Let q be an admissible integer.

Theorem (M.- D'Adderio)

1. $\chi_{\mathcal{G}}(q) = (-1)^{|\overline{V}|-k} q^k M_{\mathcal{G}}(1-q, 0)$.
2. $\chi^*_{\mathcal{G}}(q) = (-1)^{|R|-|\overline{V}|+k} q^{|D|-|V|+|\overline{V}|} M_{\mathcal{G}}(0, 1-q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

Theorem (Tutte)

1. $\chi_G(q) = (-1)^{|V|-k} q^k T_G(1-q, 0)$.
2. $\chi^*_G(q) = (-1)^{|E|-|V|+k} T_G(0, 1-q)$.

Main results

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$ be the graph obtained from $\mathcal{G} = (V, E = R \cup D)$ by (classically) contracting the edges in D. Let q be an admissible integer.

Theorem (M.- D'Adderio)

1. $\chi_{\mathcal{G}}(q) = (-1)^{|\overline{V}|-k} q^k M_{\mathcal{G}}(1-q, 0)$.
2. $\chi^*_{\mathcal{G}}(q) = (-1)^{|R|-|\overline{V}|+k} q^{|D|-|V|+|\overline{V}|} M_{\mathcal{G}}(0, 1-q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

Theorem (Tutte)

1. $\chi_G(q) = (-1)^{|V|-k} q^k T_G(1-q, 0)$.
2. $\chi^*_G(q) = (-1)^{|E|-|V|+k} T_G(0, 1-q)$.

Main results

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$ be the graph obtained from $\mathcal{G} = (V, E = R \cup D)$ by (classically) contracting the edges in D. Let q be an admissible integer.

Theorem (M.- D'Adderio)

1. $\chi_{\mathcal{G}}(q) = (-1)^{|\overline{V}|-k} q^k M_{\mathcal{G}}(1-q, 0)$.
2. $\chi^*_{\mathcal{G}}(q) = (-1)^{|R|-|\overline{V}|+k} q^{|D|-|V|+|\overline{V}|} M_{\mathcal{G}}(0, 1-q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

Theorem (Tutte)

1. $\chi_G(q) = (-1)^{|V|-k} q^k T_G(1-q, 0)$.
2. $\chi^*_G(q) = (-1)^{|E|-|V|+k} T_G(0, 1-q)$.

Main results

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$ be the graph obtained from $\mathcal{G} = (V, E = R \cup D)$ by (classically) contracting the edges in D. Let q be an admissible integer.

Theorem (M.- D'Adderio)

1. $\chi_{\mathcal{G}}(q) = (-1)^{|\overline{V}|-k} q^k M_{\mathcal{G}}(1-q, 0)$.
2. $\chi^*_{\mathcal{G}}(q) = (-1)^{|R|-|\overline{V}|+k} q^{|D|-|V|+|\overline{V}|} M_{\mathcal{G}}(0, 1-q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

Theorem (Tutte)

1. $\chi_G(q) = (-1)^{|V|-k} q^k T_G(1-q, 0)$.
2. $\chi^*_G(q) = (-1)^{|E|-|V|+k} T_G(0, 1-q)$.

Main results

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$ be the graph obtained from $\mathcal{G} = (V, E = R \cup D)$ by (classically) contracting the edges in D. Let q be an admissible integer.

Theorem (M.- D'Adderio)

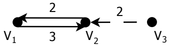
1. $\chi_{\mathcal{G}}(q) = (-1)^{|\overline{V}|-k} q^k M_{\mathcal{G}}(1-q, 0)$.
2. $\chi^*_{\mathcal{G}}(q) = (-1)^{|R|-|\overline{V}|+k} q^{|D|-|V|+|\overline{V}|} M_{\mathcal{G}}(0, 1-q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

Theorem (Tutte)

1. $\chi_G(q) = (-1)^{|V|-k} q^k T_G(1-q, 0)$.
2. $\chi^*_G(q) = (-1)^{|E|-|V|+k} T_G(0, 1-q)$.

$M_{\mathcal{G}}(x, y) = 2x + 6 + 2y$, and therefore
$\chi_{\mathcal{G}}(q) = 2q^2 - 8q$, $\chi^*_{\mathcal{G}}(q) = 2q - 8$.



Main results

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$ be the graph obtained from $\mathcal{G} = (V, E = R \cup D)$ by (classically) contracting the edges in D. Let q be an admissible integer.

Theorem (M.- D'Adderio)

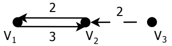
1. $\chi_{\mathcal{G}}(q) = (-1)^{|\overline{V}|-k} q^k M_{\mathcal{G}}(1-q, 0)$.
2. $\chi^*_{\mathcal{G}}(q) = (-1)^{|R|-|\overline{V}|+k} q^{|D|-|V|+|\overline{V}|} M_{\mathcal{G}}(0, 1-q)$.

When $D = \emptyset$ and $\ell \equiv 1$ we get the classical result:

Theorem (Tutte)

1. $\chi_G(q) = (-1)^{|V|-k} q^k T_G(1-q, 0)$.
2. $\chi^*_G(q) = (-1)^{|E|-|V|+k} T_G(0, 1-q)$.

$M_{\mathcal{G}}(x, y) = 2x + 6 + 2y$, and therefore
$\chi_{\mathcal{G}}(q) = 2q^2 - 8q$, $\chi^*_{\mathcal{G}}(q) = 2q - 8$.



Arithmetic quasipolynomials

And if q is not ammissible?

In general, the number of arithmetic colorings is a quasipolynomial in q.

When $q \equiv 0 \; modL(\mathcal{G})$, it coincides with the arithmetic chromatic polynomial.

When q is coprime with $L(\mathcal{G})$, it coincides with the classical chromatic polynomial.

The same for the number of arithmetic flows.

Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.

This is defined for any list X in a finitely generated abelian group Γ, in the more general language of "multivariate Tutte polynomials":

$$Q_X(q, \mathbf{v}) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{\left| q \cdot tors\left(\frac{\Gamma}{\langle A \rangle}\right) \right|} \prod_{e \in A} v_e .$$

Arithmetic quasipolynomials

And if q is not ammissible?

In general, the number of arithmetic colorings is a quasipolynomial in q.

When $q \equiv 0 \; mod L(\mathcal{G})$, it coincides with the arithmetic chromatic polynomial.

When q is coprime with $L(\mathcal{G})$, it coincides with the classical chromatic polynomial.

The same for the number of arithmetic flows.

Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial. This is defined for any list X in a finitely generated abelian group Γ, in the more general language of "multivariate Tutte polynomials":

$$Q_X(q, \mathbf{v}) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{\left| q \cdot tors\left(\frac{\Gamma}{\langle A \rangle}\right)\right|} \prod_{e \in A} v_e .$$

Arithmetic quasipolynomials

And if q is not ammissible?

In general, the number of arithmetic colorings is a quasipolynomial in q.

When $q \equiv 0 \ mod L(\mathcal{G})$, it coincides with the arithmetic chromatic polynomial.

When q is coprime with $L(\mathcal{G})$, it coincides with the classical chromatic polynomial.

The same for the number of arithmetic flows.

Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.

This is defined for any list X in a finitely generated abelian group Γ, in the more general language of "multivariate Tutte polynomials":

$$Q_X(q, \mathbf{v}) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{\left| q \cdot tors\left(\frac{\Gamma}{\langle A \rangle}\right) \right|} \prod_{e \in A} v_e.$$

Arithmetic quasipolynomials

And if q is not ammissible?

In general, the number of arithmetic colorings is a quasipolynomial in q.

When $q \equiv 0 \; modL(\mathcal{G})$, it coincides with the arithmetic chromatic polynomial.

When q is coprime with $L(\mathcal{G})$, it coincides with the classical chromatic polynomial.

The same for the number of arithmetic flows.

Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.

This is defined for any list X in a finitely generated abelian group Γ, in the more general language of "multivariate Tutte polynomials":

$$Q_X(q, \mathbf{v}) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{\left| q \cdot tors\left(\frac{\Gamma}{\langle A \rangle}\right) \right|} \prod_{e \in A} v_e .$$

Arithmetic quasipolynomials

And if q is not ammissible?
In general, the number of arithmetic colorings is a quasipolynomial in q.
When $q \equiv 0 \; modL(\mathcal{G})$, it coincides with the arithmetic chromatic polynomial.
When q is coprime with $L(\mathcal{G})$, it coincides with the classical chromatic polynomial.
The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.
This is defined for any list X in a finitely generated abelian group Γ, in the more general language of "multivariate Tutte polynomials":

$$Q_X(q, \mathbf{v}) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{\left| q \cdot tors\left(\frac{\Gamma}{\langle A \rangle}\right)\right|} \prod_{e \in A} v_e.$$

Arithmetic quasipolynomials

And if q is not ammissible?
In general, the number of arithmetic colorings is a quasipolynomial in q.
When $q \equiv 0 \; modL(\mathcal{G})$, it coincides with the arithmetic chromatic polynomial.
When q is coprime with $L(\mathcal{G})$, it coincides with the classical chromatic polynomial.
The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.
This is defined for any list X in a finitely generated abelian group Γ, in the more general language of "multivariate Tutte polynomials":

$$Q_X(q, \mathbf{v}) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{\left| q \cdot tors\left(\frac{\Gamma}{\langle A \rangle}\right) \right|} \prod_{e \in A} v_e .$$

Arithmetic quasipolynomials

And if q is not ammissible?
In general, the number of arithmetic colorings is a quasipolynomial in q.
When $q \equiv 0 \; modL(\mathcal{G})$, it coincides with the arithmetic chromatic polynomial.
When q is coprime with $L(\mathcal{G})$, it coincides with the classical chromatic polynomial.
The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.
This is defined for any list X in a finitely generated abelian group Γ, in the more general language of "multivariate Tutte polynomials":

$$Q_X(q, \mathbf{v}) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{\left| q \cdot tors\left(\frac{\Gamma}{\langle A \rangle}\right)\right|} \prod_{e \in A} v_e.$$

Arithmetic quasipolynomials

And if q is not ammissible?
In general, the number of arithmetic colorings is a quasipolynomial in q.
When $q \equiv 0 \, modL(\mathcal{G})$, it coincides with the arithmetic chromatic polynomial.
When q is coprime with $L(\mathcal{G})$, it coincides with the classical chromatic polynomial.
The same for the number of arithmetic flows.
Together with Petter Brändén, we are now studying an arithmetic Tutte quasipolynomial that specializes to the quasipolynomials above, and "interpolates" between the classical and the arithmetic Tutte polynomial.
This is defined for any list X in a finitely generated abelian group Γ, in the more general language of "multivariate Tutte polynomials":

$$Q_X(q, \mathbf{v}) := q^{rk(\Gamma)} \sum_{A \subseteq X} q^{-rk(A)} \frac{m(A)}{\left| q \cdot tors\left(\frac{\Gamma}{\langle A \rangle}\right) \right|} \prod_{e \in A} v_e.$$