Orbifold pencils and arrangements of lines

Anatoly Libgober

University of Illinois at Chicago
Chicago, Illinois

Arrangements in Pyrénées, Pau.
June 2012

Content

Content

Content

Content

References:

1.On combinatorial invariance of the cohomology of Milnor fiber of arrangements and Catalan equation over function field. arXiv:1011.0191 Arrangements of Hyperplanes, Sapporo, 2009, 175-188.

www.math.uic.edu/ libgober/mordweil

2.(with J.I.Cogolludo) Mordell-Weil groups of elliptic threefolds and the Alexander module of plane curves (To appear, Crells Journal).

3.On Mordell-Weil group of isotrivial abelian varieties over function fields.

4.(with E.Artal-Bartolo and J.I.Cogolludo) Depth of cohomology support loci for quasi-projective varieties via orbifold pencils.

5.(with E.Artal-Bartolo and J.I.Cogolludo) Characters of fundamental groups of curves complements and orbifold pencils.

There is an interesting relation between **topology** *of the complements to arrangements of lines in* $\mathbf{P}^2_{\mathbf{C}}$ *and* **geometry** *of these arrangements which helps to understnad relations between topology and* **combinatorics**.

There is an interesting relation between **topology** *of the complements to arrangements of lines in* $\mathbf{P}^2_{\mathbf{C}}$ *and* **geometry** *of these arrangements which helps to understnad relations between topology and* **combinatorics**.

Fundamental group of the complement "knows" when exist pencil of curves of degree greater than one such that several members of pencil are just unions of lines and all lines together give the arrangment.

There is an interesting relation between **topology** *of the complements to arrangements of lines in* $\mathbf{P}^2_{\mathbf{C}}$ *and* **geometry** *of these arrangements which helps to understnad relations between topology and* **combinatorics***.*

Fundamental group of the complement "knows" when exist pencil of curves of degree greater than one such that several members of pencil are just unions of lines and all lines together give the arrangment.

Including arrangement into pencil is a question about algebraic relations $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$ *where* $f_1 f_2 f_3 = \Pi l_i(x, y, z)^{n_i}$ *and* $\lambda_i \in \mathbf{C}$*.*

There is an interesting relation between **topology** *of the complements to arrangements of lines in* $\mathbf{P}^2_{\mathbf{C}}$ *and* **geometry** *of these arrangements which helps to understnad relations between topology and* **combinatorics**.

Fundamental group of the complement "knows" when exist pencil of curves of degree greater than one such that several members of pencil are just unions of lines and all lines together give the arrangment.

Including arrangement into pencil is a question about algebraic relations $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$ *where* $f_1 f_2 f_3 = \Pi l_i(x, y, z)^{n_i}$ *and* $\lambda_i \in \mathbf{C}$.

This talk: relation between fundamental group of the complement to arrangement and solutions of Diophantine equations e.g.
$f_1 X^3 + f_2 Y^3 + f_3 Z^3 = 0, X, Y, Z \in \mathbf{C}[x, y, z], f_1 f_2 f_3 = \Pi l_i(x, y, z)^{n_i}$.

Definition

Pencil of plane curves of degree d is a family
$\lambda F(x,y,z)+\mu G(x,y,z)=0$ where
$F,G\in \mathbf{C}[x,y,z], degF=degG=d, \lambda,\mu\in \mathbf{C}$

Definition

Pencil of plane curves of degree d is a family $\lambda F(x,y,z)+\mu G(x,y,z)=0$ where $F,G\in \mathbf{C}[x,y,z], degF=degG=d, \lambda,\mu\in \mathbf{C}$

Given pencil, one has a rational map $\mathbf{P}^2\to\mathbf{P}^1$ *given by* $\pi:(x,y,z)\to(F(x,y,z),G(x,y,z))$. *It is not defined at the set* $F=G=0$ *(which is finite) (if* F,G *do not have a common factor i.e. the pencil has no fixed components).*

Definition

Pencil of plane curves of degree d is a family
$\lambda F(x,y,z) + \mu G(x,y,z) = 0$ where
$F, G \in \mathbf{C}[x,y,z], degF = degG = d, \lambda, \mu \in \mathbf{C}$

Given pencil, one has a rational map $\mathbf{P}^2 \to \mathbf{P}^1$ *given by* $\pi : (x,y,z) \to (F(x,y,z), G(x,y,z))$*. It is not defined at the set* $F = G = 0$ *(which is finite) (if* F, G *do not have a common factor i.e. the pencil has no fixed components).*

Important preimages:

$$\pi^{-1}[0,1] = \{F = 0\}$$

$$\pi^{-1}[1,0] = \{G = 0\}$$

$$\pi^{-1}[1,-1] = \{F + G = 0\}$$

A pencil induces regular maps e.g.:

$$\mathbf{P}^2 - (G = 0) \to \mathbf{C}$$

given by

$$(x, y, z) \to \frac{F(x, y, z)}{G(x, y, z)}$$

Also:

$$\mathbf{P}^2 - (F = 0) \to \mathbf{C}$$

$$(x, y, z) \to \frac{G(x, y, z)}{F(x, y, z)}$$

If F_i are n members of the same pencil then one has regular map:

$$\mathbf{P}^2 - \bigcup (F_i = 0) \to \mathbf{P}^1 - \bigcup P_i$$

If F_i are n members of the same pencil then one has regular map:

$$\mathbf{P}^2 - \bigcup (F_i = 0) \to \mathbf{P}^1 - \bigcup P_i$$

Remarks

1. Any rational map $\mathbf{P}^2 \to \mathbf{P}^1$ is a pencil fo curves of some degree.
2. For any pencil exist blow up $bl : \tilde{\mathbf{P}}^2 \to \mathbf{P}^2$ such that $bl \circ \pi : \tilde{\mathbf{P}}^2 \to \mathbf{P}^1$ is a regular map.

Definition

An arrangement $\mathcal{A}$ called composed of a pencil if exist a pencils $\pi : \mathbf{P}^2 \to \mathbf{P}^1$ and points $P_1, ...P_k \in \mathbf{P}^1$ such that $\pi^{-1} P_k$ is a union of lines and $\mathcal{A} = \bigcup \pi^{-1} P_k$
Alternatively if equation of $\mathcal{A}$ is $\Pi \ell_i = 0$ (ℓ_i defined up to a constant) then exist f_1, f_2 and relations $\lambda_k f_1 + \mu_k f_2 + f_k = 0$ where $f_1 f_2 ... f_k = \Pi \ell_i^{n_i}$.

Definition

An arrangement $\mathcal{A}$ called composed of a pencil if exist a pencils $\pi : \mathbf{P}^2 \to \mathbf{P}^1$ and points $P_1, ...P_k \in \mathbf{P}^1$ such that $\pi^{-1} P_k$ is a union of lines and $\mathcal{A} = \bigcup \pi^{-1} P_k$
Alternatively if equation of $\mathcal{A}$ is $\Pi \ell_i = 0$ (ℓ_i defined up to a constant) then exist f_1, f_2 and relations $\lambda_k f_1 + \mu_k f_2 + f_k = 0$ where $f_1 f_2 ... f_k = \Pi \ell_i^{n_i}$.

Example

1. k concurrent lines $x^k - y^k = \Pi(x - \omega_k y) = 0$ form a pencil since forms $x - \omega_k y$ span a 2-dimensional space. Corresponding map: $\mathbf{P}^2 \setminus \mathcal{A} \to \mathbf{P}^1 \setminus \{k\ pts\}$

Definition

An arrangement $\mathcal{A}$ called composed of a pencil if exist a pencils $\pi : \mathbf{P}^2 \to \mathbf{P}^1$ and points $P_1, ...P_k \in \mathbf{P}^1$ such that $\pi^{-1}P_k$ is a union of lines and $\mathcal{A} = \bigcup \pi^{-1}P_k$
Alternatively if equation of $\mathcal{A}$ is $\Pi\ell_i = 0$ (ℓ_i defined up to a constant) then exist f_1, f_2 and relations $\lambda_k f_1 + \mu_k f_2 + f_k = 0$ where $f_1 f_2 ... f_k = \Pi\ell_i^{n_i}$.

Example

1. k concurrent lines $x^k - y^k = \Pi(x - \omega_k y) = 0$ form a pencil since forms $x - \omega_k y$ span a 2-dimensional space. Corresponding map: $\mathbf{P}^2 \setminus \mathcal{A} \to \mathbf{P}^1 \setminus \{k \ pts\}$

Example

2.Arrangment of $3d$ lines $(x^d - y^d)(y^d - z^d)(z^d - x^d) = 0$ form a pencil since $(x^d - y^d) + (y^d - z^d) + (z^d - x^d) = 0$

Arrangement of lines can be composed of at most five members of a pencil (Libgober-Yuzvinski)

S.Yuzvinski and J.Pereira showed that in fact there are no arrangements composed of 5 members.

Only one arrangement composed of 4 members of a pencil is known: Hesse arrangement of 12 lines containing 9 inflection points of a smooth cubic. They form 4 elements of a pencils of cubic curves.

There are no pencils of curves of degree 4,5,6 containing 4 elements composed of union of lines (Dunn, Miller, Wakefield and Zwicknagl, 2007)

Definition

Rank one local system on a topological space is a character of its fundamental group:

$$Char\pi_1(X) = CharH_1(X, \mathbf{Z}) = \mathbf{C}^{*b_1}$$

(if $TorH_1(X, \mathbf{Z}) = 0$)

Definition

Rank one local system on a topological space is a character of its fundamental group:

$$Char\pi_1(X) = CharH_1(X, \mathbf{Z}) = \mathbf{C}^{*b_1}$$

(if $TorH_1(X, \mathbf{Z}) = 0$)

Definition

Denote by $\tilde{X}_{ab}$ the universal abelian cover of X and by $\mathbf{C}_\chi$ the 1-dimensional complex space $\mathbf{C}$ with the $\pi_1(X)$-module structure given by χ. The cohomology of $\chi \in Char\pi_1(X)$ is the homology of the complex:

$$... \to C^i(\tilde{X}_{ab}) \otimes \mathbf{C}_\chi \to ...$$

$dimH^1(X, \chi)$ is called the *depth* of character.

Definition

Jumping set V_1^k is the subset of $Char\pi_1(X)$ consisting of local systems χ such that

$$dimH^1(X, \chi) \geq k$$

V_1^1 is the set of characters of positive depth. V_i^k

Definition

Jumping set V_1^k is the subset of $Char\pi_1(X)$ consisting of local systems χ such that

$$dim H^1(X,\chi) \geq k$$

V_1^1 is the set of characters of positive depth. V_i^k

Jumping loci are related to the following invariants of the fundamental group (characteristic varieties, depending only on π_1/π_1''):

$$V_1^k = Supp\Lambda^k(\pi_1'/\pi_1'') \otimes \mathbf{C} \subset Spec\mathbf{C}[\pi_1/\pi_1'] = Char\pi_1$$

Here $(\pi_1'/\pi_1'') \otimes \mathbf{C}$ is endowed with the structure of $\mathbf{C}[\pi_1/\pi_1']$-module coming from:

$$0 \to \pi_1'/\pi_1'' \to \pi_1/\pi_1'' \to \pi_1/\pi_1' \to 0$$

Theorem

(D.Arapura) Let X *be a quasi-projective manifold such that* $H^1(\bar{X}, \mathbf{Q}) = 0$ *(for example a complement to an arrangement).* $V_1^k(X)$ *are finite unions of translated by an unitary character sub-tori of* $Char\pi_1(X)$ *For each irreducible component* $\mathcal{V}$ *of* V_1^k *there exist* $\rho \in Char\pi_1(X)$ *and a holomorphic map* $f : X \to C$ *where* C *is a smooth curve such that*

$$\mathcal{V} = \rho f^*(Char\pi_1(C))$$

(ρ *has a finite order (A.Libgober))*

Remarks

1. Components of positive dimension correspond to *rational* pencils and vice versa via pull back from the target of the pencil.

Remarks

1. Components of positive dimension correspond to *rational* pencils and vice versa via pull back from the target of the pencil.

2.Let $X = \mathbf{P}^2 - \mathcal{A}$ be a complement to an arrangement. Components of positive dimension of $V_1^k \subset Char\pi_1$ containing identity, correspond to subspaces:

$$\{\alpha \in H^1(X, \mathbf{C}) | dim H^1(... \rightarrow H^i(X, \mathbf{C}) \stackrel{\wedge\alpha}{\rightarrow} H^{i+1}(X, \mathbf{C}) \rightarrow ...) \geq k\}$$

Correspondence is given by the exponential map:

$$exp : H^1(X, \mathbf{C}) \rightarrow Char\pi_1(X)$$

Remarks

1. Components of positive dimension correspond to *rational* pencils and vice versa via pull back from the target of the pencil.

2.Let $X = \mathbf{P}^2 - \mathcal{A}$ *be a complement to an arrangement. Components of positive dimension of* $V_1^k \subset Char\pi_1$ *containing identity, correspond to subspaces:*

$$\{\alpha \in H^1(X, \mathbf{C}) | dim H^1(... \to H^i(X, \mathbf{C}) \stackrel{\wedge\alpha}{\to} H^{i+1}(X, \mathbf{C}) \to ...) \geq k\}$$

Correspondence is given by the exponential map:

$$exp : H^1(X, \mathbf{C}) \to Char\pi_1(X)$$

3.In particular the components of positive dimension containing the identity are combinatorial invariants.

Orbifold pencils and local systems with non-vanishing cohomology

It follows from above that a character of **infinite** *order with* $dim H^1(\mathbf{P}^2 \setminus \mathcal{A}, \chi) \neq 0$ *is pullback via a pencil.*
What about characters of finite order?

Definition

An *orbicurve* is a complex orbifold of dimension one i.e. a smooth complex curve (compact or not) with a collection R of points (called *the orbifold points*) with a multiplicity assigned to each point in R. The complement to R is called *the regular part* of the orbifold.

Definition

An *orbicurve* is a complex orbifold of dimension one i.e. a smooth complex curve (compact or not) with a collection R of points (called *the orbifold points*) with a multiplicity assigned to each point in R. The complement to R is called *the regular part* of the orbifold.

One can think that orbicurve is a topological space locally modeled on $\mathbf{C}/\mu_n$ *where* μ_n *is a cyclic group acting on* $\mathbf{C}$ *by* $\frac{2\pi}{n}$ *rotation.*

In this viewpoint multiplicity is the order n of the group.

An orbicurve $\mathcal{C}$ *is called a global quotient if there exists a finite group G and a manifold C such that* $\mathcal{C}$ *is the quotient of C by G*

Definition

Orbifold pencil on $\mathbf{P}^2$ is a rational orbifold map $\pi : \mathbf{P}^2 \to \mathbf{P}^1_{orb}$.

In order words, orbifold pencil is a rational map $\mathbf{P}^2 \to \mathbf{P}^1$ such that for any $P \in \mathbf{P}^1$ with the property: when considered as a point in $\mathbf{P}^1_{orb}$ it has the multiplicity n, one has the following: if u is a local parameter at $P \in \mathbf{P}^1$ then the multiplicity of zero of $\pi^*(u)$ along $\pi^{-1}(P)$ divides n:

$$n | mult_{\pi^{-1}(P)} \pi^*(u)$$

Example

Consider pencil $\lambda f_1 + \mu f_2$ such that $f_1 = a^p, f_2 = b^q$ where $a, b \in \mathbf{C}[x, y, z], deg(a) = q, deg(b) = p$. The corresponding map $\mathbf{P}^2 \to \mathbf{P}^1$

$$(x, y, z) \to (a(x, y, z)^p, b(x, y, z)^q) \in \mathbf{P}^1$$

has over $[0, 1]$ a fiber of multiplicity p, and over $[1, 0]$ the fiber of multiplicity q. Hence we have a surjection onto orbifold with regular part being $\mathbf{P}^1 - [0, 1] - [1, 0]$ and $mult([0, 1]) = p, mult([1, 0]) = q$

If $F = a^p + b^q$ then one has a regular orbifold map:

$$\mathbf{P}^2 - (F = 0) \to \mathbf{C}_{p,q}$$

Remarks

One wants to consider the orbifold pencils on the complements to curves. Algebraically this corresponds to the following.

Quasi-toric relation corresponding to an orbifold pencil with target having three orbifold points is a solution in $u, v, w \in \mathbf{C}[x, y, z]$ to an equation:

$$Au^p + Bv^q + Cw^r = 0$$

where $A, B, C \in \mathbf{C}[x, y, z]$ are fixed. In the case when A, B, C are integers and one looks for integer solutions this is *Catalan* equation (Fermat is a special case).

Example

Let $f_1, f_2, f_3 \in \mathbf{C}[x, y, z]$ be forms satisfying a (quasi-toric) relation:

$$f_1 a^3 + f_2 b^3 + f_3 c^3 = 0$$

(where $a, b, c, \in \mathbf{C}[x, y, z]$). Consider rational map:

$$\mathbf{P}^2 \to \mathbf{P}^1$$

$$(x, y, z) \to (f_1 a^3, f_2 b^3)$$

Preimage of $[0, 1]$ is the union of curves $f_1 = 0, a = 0$ with fiber having multiplicity 3 along $a = 0$. Similarly for fibers over $[1, 0], [1, -1]$. If none of a, b, c has degree zero (i.e. all are non-constants) then restriction of this map onto complement to $f_1 f_2 f_3 = 0$ is surjective onto $\mathbf{P}^1$ and has at least three fibers of multiplicity 3. We have orbifold pencil:

$$\mathbf{P}^2 - (f_1 f_2 f_3) \to \mathbf{P}^1_{3,3,3}$$

Example

Example of a quasi-toric relation of type $(3,3,3)$: Consider again arrangment $\mathcal{A}$:

$$F = (y^3 - z^3)(z^3 - x^3)(x^3 - y^3) = 0$$

We have

$$x^3(y^3 - z^3) + y^3(z^3 - x^3) + z^3(x^3 - y^3) = 0$$

This yields the orbifold map $\mathbf{P}^2 - \mathcal{A} \to \mathbf{P}^1_{3,3,3}$

Example

This pencil by no means is unique (we describe below all possible pencils). We have for example:

$$\ell_1^3 F_1 + \ell_2^3 F_2 + \ell_3^3 F_3 = 0$$

where

$$F_i = (y - \omega_3^i z)(z - \omega_3^{i+1} x)(x - \omega_3^{i+2} y), \quad i = 1, 2, 3,$$

ω_3 is a third-root of unity, and

$$\begin{aligned}
\ell_1 &= (\omega_3 - \omega_3^2)x + (\omega_3 - \omega_3^2)y + (\omega_3^2 - 1)z, \\
\ell_2 &= (\omega_3 - \omega_3^2)z + (\omega_3 - \omega_3^2)x + (\omega_3^2 - 1)y, \\
\ell_3 &= (\omega_3 - \omega_3^2)y + (\omega_3 - \omega_3^2)z + (\omega_3^2 - 1)x.
\end{aligned}$$

Orbifold pencils and fundamental group

Definition

Let C be an orbicurve (compact or not) with orbifold points $P_1, ..., P_k$ with multiplicities $m_1, ..., m_k$. Let γ_i be a loop based at $P_0 \in C$ and going around P_i: it induces a conjugacy class (also γ_i) in $\pi_1(C - \bigcup P_i, P_0)$ Then

$$\pi_1^{orb}(C, P_0) = \pi_1(C - \bigcup P_i, P_0)/\gamma_i^{m_i}$$

Orbifold pencils and fundamental group

Definition

Let C be an orbicurve (compact or not) with orbifold points $P_1, ..., P_k$ with multiplicities $m_1, ..., m_k$. Let γ_i be a loop based at $P_0 \in C$ and going around P_i: it induces a conjugacy class (also γ_i) in $\pi_1(C - \bigcup P_i, P_0)$ Then

$$\pi_1^{orb}(C, P_0) = \pi_1(C - \bigcup P_i, P_0)/\gamma_i^{m_i}$$

Proposition

If $X \to C$ is an orbifold map then surjection $\pi_1(X - \pi^{-1}(\bigcup P_i)) \to \pi_1(C - \bigcup P_i))$ descents to the surjection

$$\pi_1(X) \to \pi_1^{orb}(C)$$

Corollary

Let $\pi : X \to C$ *be an orbifold map and* $\chi \in Char\pi_1^{orb}(C)$. *If* $H^1(\pi_1^{orb}(C), \chi) \neq 0$ *then* $H^1(X, \pi^*(\chi)) \neq 0$

Corollary

Let $\pi : X \to C$ be an orbifold map and $\chi \in Char\pi_1^{orb}(C)$. If $H^1(\pi_1^{orb}(C), \chi) \neq 0$ then $H^1(X, \pi^*(\chi)) \neq 0$

If C is such that $\pi_1^{orb}(C)$ has characters with $H^1(\pi_1^{orb}(C), \chi) \neq 0$ the existence of a pencil $X \to C$ with target C yields existence of local systems on X with non-vanishing cohomology. It turns out that under some assumptions one can reverse this construction but first some details on characters of orbifold fundamental groups.
To decide if for an orbifold C there are characters of $\pi_1^{orb}(C)$ with non vanishing cohomology one can use the following:

Let us denote by $\mathbf{P}^1_{m_1,\ldots,m_k,s\infty}$ *an orbicurve for which the underlying Riemann surface is* $\mathbf{P}^1$ *with* s *points removed and* k *labeled points with labels* $m_1, \ldots, m_k$. *If* $s \geq 1$ *(resp.* $s \geq 2$*) one can also use the notation* $\mathbf{C}_{m_1,\ldots,m_k,(s-1)\infty}$ *(resp.* $\mathbf{C}^*_{m_1,\ldots,m_k,(s-2)\infty}$*) for* $\mathbf{P}^1_{m_1,\ldots,m_k,s\infty}$. *Then we have:*

$$\pi_1^{orb}(\mathbf{P}^1_{m_1,\ldots,m_k,s\infty}) = \begin{cases} \mathbf{Z}_{m_1}(\gamma_1) * \ldots * \mathbf{Z}_{m_k}(\gamma_k) * F_{s-1} & \text{if } k > 0 \\ \mathbf{Z}_{m_1}(\gamma_1) * \ldots * \mathbf{Z}_{m_s}(\gamma_s)/\prod \gamma_i & \text{if } k = 0 \end{cases}$$

where $\mathbf{Z}_m(\gamma)$ *denotes a cyclic group of order* m *with a generator* γ*) and* F_{s-1} *is free group on* $s-1$ *generators.*

Usual Fox calculus is used to identify characters of π_1^{orb} *with non-vanishing cohomology.*

Example

(Orbifold pencil yields local system with non-vanishing of cohomology). For arrangement $(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$, $H^1(\mathbf{P}^2 - \mathcal{A}) = \mathbf{Z}^9/\mathbf{Z}$ (relation is $\sum \gamma_i = 0$ where γ_i is meridian about each line). Let χ be given by $\chi(\gamma_i) = \omega_3$. It is the pullback of a character of $\pi_1^{orb}(\mathbf{P}^1_{3,3,3})$ via the above pencil. Note that $Char\pi_1^{orb}(\mathbf{P}^1_{3,3,3}) = \mathbf{Z}_3^2$. It is a pull back of a character with $dimH^1 > 0$.

Next goal is to reverse contruction: finding orbifold pencils from local systems with non-vanishing cohomology.

Next goal is to reverse contruction: finding orbifold pencils from local systems with non-vanishing cohomology.

If χ is a character of infinite order then $H^1(X, \chi) \neq 0$ (here X is quasi-projective) then exist a pencil $f : X \to C$ and a finite order character ρ such that $\chi = \rho f^*(\tilde{\chi})$ where $\tilde{\chi} \in Char\pi_1(C)$. Indeed isolated characters with jumping cohomology have a finite order.

Next goal is to reverse contruction: finding orbifold pencils from local systems with non-vanishing cohomology.

If χ is a character of infinite order then $H^1(X,\chi) \neq 0$ (here X is quasi-projective) then exist a pencil $f : X \to C$ and a finite order character ρ such that $\chi = \rho f^*(\tilde{\chi})$ where $\tilde{\chi} \in Char\pi_1(C)$. Indeed isolated characters with jumping cohomology have a finite order.

In general one does not expect that characters of finite order are pullbacks of characters of orbifold fundamental groups. In the case of the complements to arrangements of curves of higher degree there are counterexamples (Artal-Cogolludo).

Also, for arrangements of curves of higher degree, one can show that isolated points of characteristic varieties are pullbacks in cases when singularities satisfy certain conditions (δ-essential, Cogolludo-L). For arrangements of lines this is the case for arrangements with double and triple points only.

Also, for arrangements of curves of higher degree, one can show that isolated points of characteristic varieties are pullbacks in cases when singularities satisfy certain conditions (δ-essential, Cogolludo-L). For arrangements of lines this is the case for arrangements with double and triple points only.

Theorem

(Cogolludo-L) Let $\mathcal{A}$ be an arrangement **with triple or double points only** *and $\chi \in Char\pi_1(\mathbf{P}^2 - \mathcal{A})$ is a character of order 3 with non vanishing cohomology. Then χ is induced by an orbifold pencil $\mathbf{P}^2 - \mathcal{A} \to \mathbf{P}^1_{3,3,3}$*

The construction of orbifold pencil from a character with non vanishing cohomology uses elliptic pencils on projective surfaces i.e. surjective holomorphic maps $X \to E$.

The construction of orbifold pencil from a character with non vanishing cohomology uses elliptic pencils on projective surfaces i.e. surjective holomorphic maps $X \to E$.

Elliptic curves with j-invariant zero and orbifolds:

Let

$$E_0 = \mathbf{C}/(a + b\omega_3)$$

$(a, b \in \mathbf{Z}, \omega_3 = exp(\frac{2\pi i}{3}))$. Multiplication of $\mathbf{C}$ by ω_3 induces automorphism T of E_0 of order 3.

T has three fixed points: $0, \frac{1+2\omega_3}{3}, \frac{2+\omega_3}{3}$.

One has:

$$E_0/(T) = \mathbf{P}^1_{3,3,3}$$

Main steps to obtain pencil from a character:
1.Consider cyclic branched covering space $W_{\mathcal{A},\chi}$ of $\mathbf{P}^2$ corresponding to the subgroup of $\pi_1(\mathbf{P}^2 - \mathcal{A})$ which is ker$\chi : \pi_1(\mathbf{P}^2 - \mathcal{A}) \to \mathbf{C}^*$. This covering space has $b_1 > 0$ (formula for the homology of cyclic branched covers of $\mathbf{P}^2$ (Libgober, 1982)).
2.There exist elliptic pencil

$$W_{\mathcal{A},\chi} \to E_0$$

where E_0 is elliptic curve with j-invariant zero. This pencil is equivariant and hence descends to an orbifold pencil $\mathbf{P}^2 \to \mathbf{P}^1_{3,3,3}$.

Main steps to obtain pencil from a character:
1.Consider cyclic branched covering space $W_{\mathcal{A},\chi}$ of $\mathbf{P}^2$ corresponding to the subgroup of $\pi_1(\mathbf{P}^2 - \mathcal{A})$ which is ker$\chi : \pi_1(\mathbf{P}^2 - \mathcal{A}) \to \mathbf{C}^*$. This covering space has $b_1 > 0$ (formula for the homology of cyclic branched covers of $\mathbf{P}^2$ (Libgober, 1982)).
2.There exist elliptic pencil

$$W_{\mathcal{A},\chi} \to E_0$$

where E_0 is elliptic curve with j-invariant zero. This pencil is equivariant and hence descends to an orbifold pencil $\mathbf{P}^2 \to \mathbf{P}^1_{3,3,3}$.

Construction of elliptic pencils is radically different than construction of deFranchis.... and uses structure theorem for Albanese varieties of cyclic covers.

Definition

Let W be a smooth projective variety. There exist universal abelian variety $Alb(W)$ and map $w : W \to Alb(W)$ such that any map $f : W \to A$ factors through $Alb(W)$ i.e. exist $w' : Alb(W) \to A$ such that $f = w' \circ w$

$$Alb(W) = H^0(W, \Omega^1_W)^* / H_1(W, \mathbf{Z})$$

where $H_1(W, \mathbf{Z})$ is a subgroup of $H^0(W, \Omega^1_W)^*$ via the embedding

$$\gamma \Rightarrow (\omega \to \int_\gamma \omega)$$

Definition

Let W be a smooth projective variety. There exist universal abelian variety $Alb(W)$ and map $w: W \to Alb(W)$ such that any map $f: W \to A$ factors through $Alb(W)$ i.e. exist $w': Alb(W) \to A$ such that $f = w' \circ w$

$$Alb(W) = H^0(W, \Omega^1_W)^* / H_1(W, \mathbf{Z})$$

where $H_1(W, \mathbf{Z})$ is a subgroup of $H^0(W, \Omega^1_W)^*$ via the embedding

$$\gamma \Rightarrow (\omega \to \int_\gamma \omega)$$

Theorem

Let W be a cyclic cover of $\mathbf{P}^2$ branched over an arrangement with double and triple points only. Then $Alb(W)$ is isomorpic to a product of elliptic curves E_0 with j-invariant zero.

Remarks

1.For an arrangment with with double and triple points, 3-fold cyclic cover admits elliptic pencil $W \to Alb(W) = E_o^k \to E_0$. This pencil is equivariant and hence induces an orbifold pencil.

Remarks

1.For an arrangment with with double and triple points, 3-fold cyclic cover admits elliptic pencil $W \to Alb(W) = E_o^k \to E_0$. This pencil is equivariant and hence induces an orbifold pencil.

2.For arrangements with double and triple points among isolated characters such that $\chi(\gamma_i) = \chi(\gamma_j)$ only characters of order 3 can satisfy dim$H^1(\mathbf{P}^2 - \mathcal{A}, \chi) > 0$. This follows from relation between the Alexander polynomials and homology of cyclic covers and properties of Alexander polynomials (A.Libgober)

Remarks

1.For an arrangment with with double and triple points, 3-fold cyclic cover admits elliptic pencil $W \to Alb(W) = E_o^k \to E_0$. This pencil is equivariant and hence induces an orbifold pencil.

2.For arrangements with double and triple points among isolated characters such that $\chi(\gamma_i) = \chi(\gamma_j)$ only characters of order 3 can satisfy dim$H^1(\mathbf{P}^2 - \mathcal{A}, \chi) > 0$. This follows from relation between the Alexander polynomials and homology of cyclic covers and properties of Alexander polynomials (A.Libgober)

3. Vice versa, for any smooth projective variety with elliptic pencil $W \to E$ by Poincare reducibility theorem $Alb(W)$ is isogenous to product $E \times A$.

4.If $k > 1$ the projection is non unique i.e. yields multiple quasi-toric relations:

$$f_1 A_1^3 + f_2 A_2^3 + f_3 A_3^3 = 0$$

Note that if $k = 1$ then $Hom(E_0, E_0) = \mathbf{Z}[\omega_3]$. In general pencils form a group isomorphic to $Hom_{\mu_3}(E_0^k, E_0)$ (the group of equivariant maps).

4.If $k > 1$ the projection is non unique i.e. yields multiple quasi-toric relations:

$$f_1 A_1^3 + f_2 A_2^3 + f_3 A_3^3 = 0$$

Note that if $k = 1$ then $Hom(E_0, E_0) = \mathbf{Z}[\omega_3]$. In general pencils form a group isomorphic to $Hom_{\mu_3}(E_0^k, E_0)$ (the group of equivariant maps).

5.For Hesse arrangement $\mathcal{H}$ of 12 lines containing inflection points of a smooth cubic the Albanese variety of double cover ramified along $\mathcal{H}$ is also E_0. The reason: cross ration of quadruple point.

6.Relation between decomposibility of Albanese of cyclic covers of $\mathbf{P}^2$ *and local type of singularities valid for arbitrary plane algebraic curves: there is condition on singularities (local Albanese variety has complex multiplication of certain type) which implies that Albanese of cyclic covers is isogenous to a product of simple abelian varities of CM type. (A.Libgober)*

Orbifold pencils corresponding to order two characters and having as target $\mathbf{C}_{2,2}$: global quotient of $\mathbf{C}$ by involution $z \to z^{-1}$.
Corresponding quasitoric relation has form:

$$fU^2 + gV^2 = F_\chi$$

where F_χ is product of forms corresponding to lines $\ell_i = 0$ such that $\chi(\gamma_{\ell_i}) = -1$ and $F_\chi = fg$.
If χ is a pull back of the character of $\pi_1^{orb}(\mathbf{C}_{2,2})$ which does not extends to $\mathbf{P}^1_{2,2}$ χ is essentail and coordinate. Vice versa, given such a quasi-toric relation one obtains essential coordinate character.

Definition

Coodinate characters: A character $\chi \in \mathit{Char}(\pi_1(\mathbf{P}^2 - \mathcal{A}))$ is called coordinate if there is a subarrangement $\mathcal{A}'$ and $\chi' \in \mathit{Char}\pi_1(\mathbf{P}^2 - \mathcal{A}')$ such that for canonical embedding $i : \mathbf{P}^2 - \mathcal{A} \to \mathbf{P}^2 - \mathcal{A}'$ one has $\chi = i^*(\chi')$.

Definition

Coodinate characters: A character $\chi \in Char(\pi_1(\mathbf{P}^2 - \mathcal{A}))$ is called coordinate if there is a subarrangement $\mathcal{A}'$ and $\chi' \in Char\pi_1(\mathbf{P}^2 - \mathcal{A}')$ such that for canonical embedding $i : \mathbf{P}^2 - \mathcal{A} \to \mathbf{P}^2 - \mathcal{A}'$ one has $\chi = i^*(\chi')$.

Definition

Essential characters: A character $\chi \in Char(\pi_1(\mathbf{P}^2 - \mathcal{A}))$ is called essential if $dimH^1(\chi) > 0$ and there is no subarrangement $\mathcal{A}' \subset \mathcal{A}$ and $\chi' \in Char\pi_1(\mathbf{P}^2 - \mathcal{A}')$, $dimH^1(\chi') > 0$ such that for canonical embedding $i : \mathbf{P}^2 - \mathcal{A} \to \mathbf{P}^2 - \mathcal{A}'$ one has $\chi = i^*(\chi')$.

Definition

Coodinate characters: A character $\chi \in Char(\pi_1(\mathbf{P}^2 - \mathcal{A}))$ is called coordinate if there is a subarrangement $\mathcal{A}'$ and $\chi' \in Char\pi_1(\mathbf{P}^2 - \mathcal{A}')$ such that for canonical embedding $i : \mathbf{P}^2 - \mathcal{A} \to \mathbf{P}^2 - \mathcal{A}'$ one has $\chi = i^*(\chi')$.

Definition

Essential characters: A character $\chi \in Char(\pi_1(\mathbf{P}^2 - \mathcal{A}))$ is called essential if $dimH^1(\chi) > 0$ and there is no subarrangement $\mathcal{A}' \subset \mathcal{A}$ and $\chi' \in Char\pi_1(\mathbf{P}^2 - \mathcal{A}')$, $dimH^1(\chi') > 0$ such that for canonical embedding $i : \mathbf{P}^2 - \mathcal{A} \to \mathbf{P}^2 - \mathcal{A}'$ one has $\chi = i^*(\chi')$.

There is a Hodge theretical characterization of such characters as characters having weight 2 only (Artal-Cogolludo-L).

The proof of correspondence between such characters and orbifold pencils $\mathbf{C}_{2,2}$ *involves study of Albanese variety of quasi-projective manifolds (semi-abelian varieties)*

Problem:

Are the characteristic polynomials of the monodromy acting on the cohomology groups of Milnor fiber of an arrangmenent, combinatorial invariants?

Problem:

Are the characteristic polynomials of the monodromy acting on the cohomology groups of Milnor fiber of an arrangmenent, combinatorial invariants?

For an arrangement with double and triple points only such characteristic polynomial has form

$$(t-1)^{\#\mathcal{A}-1}(t^2+t+1)^{k(\mathcal{A})}$$

so the problem is the combinatorical invariance of $k(\mathcal{A})$. One can get a weaker version of combinatorial invariants using orbifold pencils as above using the following result.

Theorem

If Catalan equation $f_1A_1^3 + f_2A_2^3 + f_3A_3^3 = 0$, $f_1 \cdot f_2 \cdot f_3 = \Pi \ell_i^{k_i}$ *corresponding to an arrangement with double and triple points has a solution then it has a* **constant** *solution.*

Theorem

If Catalan equation $f_1 A_1^3 + f_2 A_2^3 + f_3 A_3^3 = 0$, $f_1 \cdot f_2 \cdot f_3 = \Pi \ell_i^{k_i}$ *corresponding to an arrangement with double and triple points has a solution then it has a* **constant** *solution.*

Corollary

If two arrangmenent are combinatorially equivalent and if monodromy of Milnor fiber for one of them has positive multiplicity of the root ω_3 *then so is the monodomy of Milnor fiber for another.*

Problem: how big this multiplicity can be?

Problem: how big this multiplicity can be?

For $\mathcal{A}$*:* $(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$ *one has* $k(\mathcal{A}) = 2$*. Are there bigger values?*

Problem: how big this multiplicity can be?

For $\mathcal{A}$: $(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$ one has $k(\mathcal{A}) = 2$. Are there bigger values?

There are other interesting connections between depth (or $k(\mathcal{A})$) and geometry.
1. $k(\mathcal{A})$ is related to the Mordell-Weil ranks of elliptic threefolds associated with $\mathcal{A}$. If ranks of Mordell-Weil groups are bounded then so are $k(\mathcal{A})$.
2.Number of pencils of which arrangement can be composed. for $\mathcal{A}$ this is 4. If one can show that the number of pencils inducing $\chi : \chi(\gamma_i) = \omega_3$ determines and is determined by $k(\mathcal{A})$ then one can get combinatorial invariance of $k(\mathcal{A})$.