New poles of the Igusa local zeta function in positive characteristic

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Notations

Let

- K be a non-archimedean local field;
- $\mathcal{O}_K$ be the ring of integers of K and m_K its maximal ideal;
- $\overline{K} = \mathcal{O}_K / m_K$ be the residue field of K having the cardinality $q = p^r$, where p is a prime;
- π be a uniformizing parameter of K;
- v denote the valuation on K such that $v(\pi) = 1$;
- for $x \in K$, let $|x| = q^{-v(x)}$.

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The Igusa local zeta function

Definition

Let $f(x) = f(x_1, \ldots, x_n) \in \mathcal{O}_K[x_1, \ldots, x_n]$. The **Igusa local zeta function** associated to f is

$$Z_f(s) = \int_{(x_1,\ldots,x_n)\in\mathcal{O}_K^n} |f(x_1,\ldots,x_n)|^s |dx_1| \ldots |dx_n|,$$

where $s \in \mathbb{C}$, $Re(s) > 0$, $|dx| = |dx_1| \cdot \ldots \cdot |dx_n|$ denotes the Haar measure on K^n so normalized that $\mathcal{O}_K^n$ has measure 1.

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The rationality of the Igusa local zeta function

- If $char(K) = 0$, then $Z_f(s)$ is a rational function.
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The Poincaré series

- Igusa's zeta function is directly related to the number of solutions of the congruences $f(x) \equiv 0 \mod \pi^e \mathcal{O}_K$, for $e \in \mathbb{N}$, that are codified by the Poincaré series.
- For $f(x) \in \mathcal{O}_K[x]$, let

$$M_e = \# \left\{ x \in (\mathcal{O}_K/\pi^e\mathcal{O}_K)^n \,|\, f(x) \equiv 0 \mod \pi^e\mathcal{O}_K \right\},$$

for $e \geq 1$ and $M_0 := 1$;
- Let $P_f(t) = \sum_{e=0}^{\infty} M_e q^{-ne} t^e$ be the Poincaré series of f;

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The connection with the Poincaré series

- There is the relation

$$P_f(q^{-s}) = \frac{1 - q^{-s} Z_f(s)}{1 - q^{-s}}.$$

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Motivation

Why is it interesting to study the Igusa local zeta function?

- Connection with the Poincaré series;
- Poles of the Igusa zeta function determine the asymptotic behavior of the number of solutions of polynomial congruences;
- Monodromy conjecture (in characteristic zero).

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The subject of the present work

- We give a complete description of the poles of the Igusa zeta function of some classes of polynomials, called hybrid polynomial, by determining its explicit form.

Definition

Let K be a non-archimedean local field of characteristic p, with p a prime number.

Definition

*A **hybrid polynomial** is a polynomial having the form*

$$f(x,y,z) = x^p + y^r z^s \sum_{i=0}^{k} \binom{k+r}{i+r} y^i (tz-y)^{k-i} \in K[x,y,z],$$

where

Definition

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Where

- k *is a positive integer;*
- r *and* s *are positive integers not divisible by* p *such that* $r + s + k$ *is a multiple of the characteristic* p*;*
- *the residues* $\bar{r}$ *and* $\bar{s}$ *of* r *and* s *modulo* p *satisfies* $\bar{r} + \bar{s} \leq p$*;*
- t *is an arbitrary constant from the ground field* K*.*

Why hybrid polynomials?

- Introduced by Hauser, in 2010.
- They appeared naturally, when he studied completely the cases in characteristic p where the arguments and conclusions of the proof of the existence of the resolution of singularities in characteristic 0 fail.
- In the case of polynomials in three variables, he observed that all these obstructions only occur in concrete series of polynomials, the hybrid polynomials.

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Why hybrid polynomials?

- They describe completely the hypersurface singularities in characteristic p where the characteristic zero proof of the embedded resolution of singularities fails.
- The rationality of the ILZF is a consequence of the existence of an embedded resolution of singularities.

Why is it interesting to determine the ILZF of hybrid polynomials?

- A description of the poles of Z_f can be obtained by using an embedded resolution of singularities of f.
- 2009, Cossart and Piltant - an embedded resolution of singularities exists in the case of surfaces in $\mathbb{A}^3_K$ in positive characteristic.

The relevance of the problem

What is it already known:

- This implies that the Igusa zeta function of hybrid polynomials will be a rational function.
- If one has an explicit embedded resolution, this will also give a list of candidate poles of the Igusa zeta function.

In practice,

- for a given polynomial, it is difficult to find an explicit embedded resolution;
- even if one can find it, the list of candidate poles given by the numerical invariants associated to such an embedded resolution gives a very long list of candidate poles.

Our work provides a list with just three candidate poles.

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Main theorem

Theorem (Cardenal, I., Segers, 2012)

Let $f(x,y,z) = x^p + yz\sum_{i=0}^{k}\binom{k+1}{i+1}y^i(tz-y)^{k-i} \in K[x,y,z]$, *where* k *is a positive natural number such that* $p|k+2$ *and* t *is a constant in the ground field* K *such that* t^{k+1} *is a* $p-$*th power of an element of* $\mathcal{O}_K^\times$.

Main theorem

Theorem (Cardenal, I., Segers, 2012)

Then the Igusa local zeta function of $f(x, y, z)$ is a rational function of q^{-s} of the form $Z_f(s) =$

$$= \frac{L(q^{-s})}{\left(1 - q^{-1-s}\right)\left(1 - q^{-((k+2)/p)-2-(k+2)s}\right)\left(1 - q^{-p-k-1-p(k+1)s}\right)},$$

where $L(q^{-s})$ is a polynomial with rational coefficients which can be computed effectively.

Corollary

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Let $f(x,y,z) = x^p + yz\sum_{i=0}^{k}\binom{k+1}{i+1}y^i(tz-y)^{k-i} \in K[x,y,z]$, *with the conditions on* k *and* t *given in the previous theorem. Then the real parts of the candidate poles of the Igusa local zeta function of* f *are* -1, $-\frac{1}{p}-\frac{2}{k+2}$ *and* $-\frac{1}{p}-\frac{1}{k+1}$.

Comments

- The class of hybrid polynomials that we consider is an infinite class;
- We get the two candidate poles that we have expected, $-1, -\frac{1}{p} - \frac{2}{k+2}$ which are connected with the degree of the polynomial and with its weight.
- A new candidate pole appeared: $-\frac{1}{p} - \frac{1}{k+1}$, which is related with the characteristic of the ground field.