(Partial) formality.

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A path-connected space whose minimal model is isomorphic to the minimal model of its cohomology ring is called formal(Sullivan).

Compact Kähler manifolds (in particular, smooth complex projective varieties) are important examples of formal spaces (Deligne-Morgan-Griffiths-Sullivan).

Rational cohomology spheres or complements of complex hyperplane arrangements are also examples of formal spaces (Brieskorn).

Eilenberg-McLane spaces $K(\pi, n)$ with $n \geq 2$ are again formal.

1-formality is a classical weaker property, depending only on the fundamental group of the space. It is originally defined as the quadratic presentability of the Malcev Lie algebra associated to the fundamental group.

We introduce and investigate k-stage formality. For $k = 1$, it coincides with the classical 1-formality and for $k = \infty$ we get full formality in the sense of Sullivan.

An interesting notion of partial formality was introduced more recently by Fernández and Muñoz. This new notion is different from the classical one for $k = 1$, and it is strictly stronger than k-stage formality.

It is well-known that 1-formality is the first general obstruction in the Serre problem regarding the characterization of *projective groups* (fundamental groups of smooth projective complex varieties).
A difficult particular case of this problem turns out to be that of nilpotent groups.

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Cap. I (Partial) minimal models. (Partial) formality.

Let (A^*, d_A) be a differential graded commutative algebra (D.G.A.) over a field $\Bbbk$ of characteristic zero such that $H^0(A^*, d_A)$ is the ground field. A minimal model for A^* is a minimal D.G.A. $(\mathcal{M}, d_{\mathcal{M}})$ such that there exists a morphism of D.G.A.'s $\rho : \mathcal{M} \longrightarrow A^*$ inducing a cohomology isomorphism.

Up to isomorphism, there is a unique minimal model, $\mathcal{M} = \mathcal{M}(A)$.

The minimal model of a space K having the homotopy type of a connected simplicial complex K, denoted by $\mathcal{M}(K)$, is the minimal model associated to the D.G.A. of P.L. forms $\Omega^*(K)$.

A space K is formal if the minimal model of K coincides with the minimal model of its cohomology.

Equivalently, K is formal if there is a sequence of D.G.A. morphisms connecting $\Omega^*(K)$ to $(H^*(K), 0)$, not necessarily going all in the same direction, such that each of them induces an isomorphism in cohomology.

A minimal algebra $\mathcal{M}$ generated by elements of degree $\leq k$ is a *k-minimal model* of a D.G.A. (A^*, d_A) if there exists a D.G.A. map $\rho : \mathcal{M} \longrightarrow A$ such that ρ induces in cohomology isomorphisms up to degree k and a monomorphism in degree $k+1$. Up to isomorphism, there is a unique k-minimal model, $\mathcal{M} = \mathcal{M}_k(A)$.

A D.G.A. (A^*, d_A) is called *k- stage formal* if there is a sequence of D.G.A. morphisms connecting (A^*, d_A) to $(H^*(A), 0)$, not necessarily going all in the same direction, such that each of them induces an isomorphism in cohomology up to degree k and a monomorphism in degree $k+1$. A space K is called *k- stage formal* if the D.G.A. $\Omega^*(K)$ is k-stage formal.

The 1-formality property of a space K depends only on the fundamental group of that space. Define the 1-minimal model of a group to be the 1-minimal model of the Eilenberg-McLane space $K(G, 1)$.

It is well known that formality of a space implies the 1-formality of the fundamental group of that space.

Cap. II Obstructions to partial formality.

Given a non-negatively graded algebra H^*, we will denote by $H^{\leq k+1}$ the quotient algebra $H^*/H^{>k+1}$ obtained by truncation up to degree $k+1$.

For a finitely generated nilpotent group, we develop obstructions to k-stage formality involving either the generators of the truncated cohomology ring or certain resonance varieties associated to the cohomology ring.

Theorem II.1 (M)

Let G be a finitely generated nilpotent group.

1. If G is k-stage formal, then $H^{\leq k+1}(G)$ is generated as an algebra by $H^1(G)$.
2. For a 2-step nilpotent group, the converse holds as well.

For $k = 1$, the first part of the theorem was also proved by Amorós-Burger-Corlette-Kotschick-Toledo and the second part follows from a result of Carlson-Toledo.

Proposition (M.) A k-formal space M with $H^{\leq k+2}(M) = 0$ is formal.

Corollary Complex plane projective curve complements are formal spaces.
This result was also proved by Cogolludo-Matei, using different methods.

(Falk) Let H^* be a connected graded-commutative $\mathbb{k}$-algebra. The resonance variety

$$\mathcal{R}_k^q(H^*) = \{\omega \in H^1 | \ \dim_{\mathbb{k}} H^q(H^*, \mu_\omega) \geq k\}$$

where μ_w is the differential given by left-multiplication by w in H^*; this is a homogeneous algebraic subvariety of the affine space H^1, when H^* is of finite type as a graded vector space. We denote

$\mathcal{R}_k^q(H^*)$ by $\mathcal{R}_k^q(M)$ when M is a path-connected space and $H^* = H^*(M, \mathbb{k})$; for $M = K(G, 1)$ we use the notation $\mathcal{R}_k^q(G)$.

Theorem II.2 (M.) The resonance varieties of a finitely generated, nilpotent s-stage formal group G are trivial up to degree s, that is, $\mathcal{R}_1^i(G) \subseteq \{0\}$ for $i \leq s$.

The fundamental group of the complement of a complex hyperplane arrangement is 1-formal. It turns out that the nilpotency test from the above Theorem, via resonance, is faithful for this class of groups.

Let $G_{\mathcal{A}} = \pi_1(M_{\mathcal{A}})$ be the fundamental group of the complement of a central complex hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^n$, $n \geq 3$.The following properties are equivalent:

1. The hyperplanes of $\mathcal{A}$ are in general position in codimension 2.
2. The group $G_{\mathcal{A}}$ is abelian.
3. The group $G_{\mathcal{A}}$ is nilpotent.
4. $\dim_{\mathbb{Q}} \mathrm{gr}(G_{\mathcal{A}}) \otimes \mathbb{Q} < \infty$.
5. $\mathcal{R}_1^1(G_{\mathcal{A}}) \subseteq \{0\}$.
6. $\mathcal{V}_1^1(G_{\mathcal{A}}) \subseteq \{1\}$.

(Here $\mathrm{gr}(G) \otimes \mathbb{Q}$ is the rational graded Lie algebra associated to the lower central series of G, and $\mathcal{V}_1^1(G)$ denotes the first characteristic variety of G in degree one.)

Cap. III Heisenberg (type) groups.

The Heisenberg group $\mathcal{H}_n$ is given by the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_n \longrightarrow \mathbb{Z}^{2n} \longrightarrow 0, \tag{1}$$

corresponding to the cohomology class
$\omega \in H^2(\mathbb{Z}^{2n}, \mathbb{Z}) = \wedge^2_{\mathbb{Z}}(x_1, y_1, \dots, x_n, y_n)$, where
$\omega = x_1 \wedge y_1 + \cdots + x_n \wedge y_n$.

Let G be a finitely generated, 2-step nilpotent group defined by a central extension of the form

$$0 \longrightarrow B \longrightarrow G \longrightarrow A \longrightarrow 0, \tag{2}$$

where B is an abelian group of rank 1, and A is an abelian group of finite rank m. The minimal model of G is then of the form
$\mathcal{M}(G) = \wedge(t_1, \dots, t_m) \otimes \wedge(z)$, with differential given by
$d(t_i) = 0$, $\forall i$ and $d(z) := \omega \in \wedge^2(t_1, \dots, t_m)$.

The group G is called of Heisenberg type if $\omega \neq 0$.

For this family of groups, the obstructions to partial formality from theorems II.1 and II.2 coincide.

The Heisenberg group $\mathcal{H}_n$ is $(n-1)$-stage formal, but not n-stage formal, and the degree of partial formality is precisely detected by the resonance for Heisenberg-type groups, via the test provided by Theorem II.2.

Returning to the before mentioned question of Serre, if $n = 2, 3$ the groups $\mathcal{H}_n$ are not projective, as shown by Carlson-Toledo and $\mathcal{H}_1$ is not even 1-formal.
Campana realised $\mathcal{H}_{n\geq 4}$ as projective groups. We prove that the varieties involved in his solution for $\mathcal{H}_n$ must have a nontrivial homotopy group π_i with $2 \leq i \leq n$.

This presentation is based on the paper M., Cohomology rings and formality properties of nilpotent groups, Journal of Pure and Applied Algebra 214 (2010), pp. 1818-1826.