On the number of complement regions in submanifold arrangements

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Let us consider m–dimensional manifold M and the finite family $\{A_1, \ldots, A_n\}$ of closed $(m-1)$–dimensional subsets. Let

$$f = |\pi_0(M \setminus \bigcup_{i=1}^{n} A_i)|$$

be the connected components number of the complement in M to the union of $A_1, \ldots, A_n$. Let $F_n(M)$ be the set of numbers f for all possible arrangements of n subsets given type. The general question is to describe the sets $F_n(M)$ for arrangements of closed geodesics or totally geodesic surfaces in M. The sets $F_n(M)$ could be interesting in connection with Orlic and Solomon [1] statement that the region number in hyperplane arrangements equals to the cohomology ring dimension of the complement to complexified arrangement. N. Martinov [2] founded the sets of region numbers in real projective plane arrangements of lines and arrangements of pseudolines. Hence the sets of region numbers in standard sphere arrangements of big circles are also known.

Theorem. Let us consider arrangements of $(m-1)$–dimensional flat subtori in the m–dimensional flat torus T^m, arrangements of closed (non–simple) geodesics in the flat Klein bottle KL^2, arrangements of closed simple geodesics in the surface R of the tetrahedron, arrangements of hyperplanes in the hyperbolic metric of Lobachevsky space L^m and finally hyperplane arrangements with empty intersection of all hyperplanes in the real projective space P^m. Then

$$F_n(T^m) \supseteq \{n-m+1, \ldots, n\} \cup \{l \in \mathbb{N} \mid l \geq 2(n-m)\} \qquad (1)$$

$$F_n(KL^2) = \{n+1\} \cup F_n(T^2) \quad \text{for } n \geq 2, \; F_1(KL^2) = \mathbb{N},$$

$$F_n(R) \subseteq \{n+1, 2n\} \cup \{l \in \mathbb{N} \mid l \geq 4n-6\} \quad \text{for } n \geq 3, \qquad (2)$$

$$F_n(L^m) = \left\{ f \in \mathbb{N} \mid n+1 \leq f \leq \sum_{i=0}^{m} \binom{n}{i} \right\}$$

first four numbers of $F_n(P^m)$ for $n \geq 2m+5$ and $m \geq 3$ are:

$$(n-m+1)2^{m-1}, 3(n-m)2^{m-2}, (3n-3m+1)2^{m-2}, 7(n-m)2^{m-3}.$$

The inclusion (1) turns into equality at least for two–dimensional tori. The inclusion (2) turns into equality iff all tetrahedron faces are equal acute–angled triangles.

References.

[1] P. Orlic, L. Solomon, Combinatorics and topology of complements of hyperplanes. // Inventiones Math. **50** (1980), 167 – 189.

[2] N. Martinov, Classification of arrangements by the number of their cells // Discr. and Comput. Geom., **9** : 1 (1993), 39 – 46.

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