Normality and Minkowski sum of Lattice Polytopes

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Lattice Polytopes

Let $M = \mathbb{Z}^n$ be a free abelian group of rank n and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ the extension of coefficients into real numbers. We define a *lattice polytope* as the convex hull $P := \text{Conv}\{m_1, \ldots, m_r\}$ of a finite subset $\{m_1, \ldots, m_r\}$ of M in $M_{\mathbb{R}}$. We define the dimension of a lattice polytope P as that of the smallest affine subspace containing P.

Definition[1, Definition 2.59] A lattice polytope P in $M_{\mathbb{R}}$ is called *normal* if the equality

$$\overbrace{(P \cap M) + \cdots + (P \cap M)}^{k \text{ times}} = (kP) \cap M \qquad (1)$$

holds for all positive integers k. The quality (1) is equivalent that for each lattice point $v \in (kP) \cap M$ there exists just k elements $u_1, \ldots, u_k \in P \cap M$ such that $v = u_1 + \cdots + u_k$.

Definition[1, Exercise 2.23] A lattice polytope P in $M_{\mathbb{R}}$ is called *very ample* if for each vertex v of P the affine monoid $C_v(P) \cap M$ of the cone

$$C_v(P) := \mathbb{R}_{\geq 0}(P - v) = \{r(x - v) \in M_{\mathbb{R}}; r > 0 \text{ and } x \in P\}$$

is generated by $(P - v) \cap M$.

Definition For a lattice polytope P, we define the associated graded $\mathbb{C}$-algebra

$$R(P) := \mathbb{C} \oplus \bigoplus_{k=1}^{\infty} \langle (kP) \cap M \rangle_{\mathbb{C}}.$$

Remark 1 In terms of toric variety in algebraic geometry, a lattice polytope P corresponds to a polarized toric variety (X, L) so that the vector space $\langle (kP) \cap M \rangle_{\mathbb{C}}$ coincides with the space of global sections $\Gamma(X, L^{\otimes k})$. See Fulton's book [4], or Oda's book [6]. P is normal if and only if L is normally generated, and P is very ample if and only if L is very ample.

Remark 2 If P is normal, then $R(P)$ is generated by elements of degree one, that is, $R(P)$ is normal ring. In contrast, if P is very ample, then the part of $R(P)$ with large degree is generated by degree one elements.

Ewald and Wessels [3] showed that the k times multiple kP of a lattice polytope P of dimension n is very ample for $k \geq n - 1$. Nakagawa [5] showed that the equality

$$(kP) \cap M + (P \cap M) = ((k + 1)P) \cap M$$

holds for $k \geq n - 1$. Hence kP is normal for $k \geq n - 1$.

Theorem 1 [7] *For $n \geq 3$, there exists a lattice polytope P of dimension n such that P is very ample but $(n - 2)P$ is not normal.*

Remark 3 We know only two examples such that it is very ample but not normal, one is in dimension three [1] and the other is in dimension five [2].

Not Very Ample Lattice Polytopes

Set $n \geq 3$. Let $\{e_1, \ldots, e_n\}$ be a $\mathbb{Z}$-basis of M. For a positive integer q, set $v_0 = 0$, $v_i = e_i$ for $i = 1, \ldots, n - 1$ and $v_n = e_1 + \cdots + e_{n-1} + qe_n$. Set the n-simplex as

$$\Delta_q^n := \text{Conv}\{v_0, v_1, \ldots, v_n\},$$

and the basic n-simplex as

$$\Delta_0^n := \text{Conv}\{0, e_1, \ldots, e_n\}.$$

We note that the basic n-simplex is normal, hence, it is very ample.

Proposition 1 *For $q \geq 2$, the lattice n-simplex Δ_q^n is not very ample. Moreover, if $q \geq n - 1$, then the multiple $(n - 2)\Delta_q^n$ is not very ample.*

Proof. Set $u = e_1 + \cdots + e_n$. Then the lattice point u is contained in the cone $C_{v_0}(\Delta_q^n)$. We note that u is the nearest to the origin among lattice points in the cone with the coefficient of e_n one. On the other hand, we see that u is not contained in $(n-2)\Delta_q^n$ for $q \geq n-1$. This implies that u cannot be represented as sum of lattice points in $(n - 2)\Delta_q^n$.

Minkowski Sum

For two lattice polytopes P and Q in $M_{\mathbb{R}}$, we define the *Minkowski sum* of P and Q as

$$P + Q := \{x + y \in M_{\mathbb{R}}; x \in P \text{ and } y \in Q\}.$$

Set $n = 3$ and $M = \mathbb{Z}^3$. We identify $M_{\mathbb{R}}$ as $\mathbb{R}^3$ with the coordinates (x, y, z). For an integer $q \geq 1$, set the 3-simplex

$$\Delta_q := \text{Conv}\{0, (1, 0, 0), (0, 1, 0), (1, 1, q)\} \subset \mathbb{R}^3$$

and the line segment

$$I := \text{Conv}\{0, (0, 0, 1)\}.$$

When $q = 4$, the Minkowski sum $\Delta_q + I$ coincides with the example of [1] up to an affine transformation of M. It is easy to see that $\Delta_q + I$ is very ample but not normal when $q \geq 4$.

Higher Dimension

In this section we set $n \geq 3$. Let $\{e_1, \ldots, e_n\}$ be a $\mathbb{Z}$-basis of M. Set the line segment as

$$I := \text{Conv}\{0, e_n\}.$$

Then we define a lattice polytope P_q as the Minkowski sum

$$P_q := \Delta_q + I. \qquad (2)$$

Proposition 2 *Let P_q be the lattice polytope defined by* (2). *Then P_q is very ample.*

Proposition 3 *When $n \geq 4$, if $q \geq 2$ then P_q is not normal.*

Proposition 4 *When $n \geq 3$, if $q \geq 2(n - 1)(n - 2)$ then $(n - 2)P_q$ is not normal.*

Three Dimension

In this section we set $n = 3$, and we will give examples of normal lattice polytope which is the Minkowski sum of a lattice 3-simplex and a line segment.

For integers $0 \leq p < q$ define the 3-simplex

$$\Delta_{p,q} := \text{Conv}\{0, (1, 0, 0), (0, 1, 0), (1, p, q)\}.$$

When $q = 1$ this lattice simplex coincides with the basic 3-simplex Δ_0^3. The terminal lemma [6, p.34] says that if a lattice polytope of dimension three contains only four lattice points, then it is isomorphic to $\Delta_{p,q}$ up to an affine transformation of M. We note that if $q \geq 2$, then $p \geq 1$ and $\text{g.c.d.}(p, q) = 1$.

Let (x, y, z) be a coordinate system of $M_{\mathbb{R}} \cong \mathbb{R}^3$. The lattice polytope $\Delta_{p,q}$ is contained in the parallel reagion $\{0 \leq x \leq 1\}$, and it is also the convex hull of two line segments $A' := \Delta_{p,q} \cap (x = 0)$ and $B' := \Delta_{p,q} \cap (x = 1)$. In the yz-plane set

$$A = \text{Conv}\{0, (1, 0)\}, \quad B = \text{Conv}\{0, (p, q)\}.$$

Then we have $A' = 0 \times A$ and $B' = 1 \times B$ in $\mathbb{R} \times \mathbb{R}^2$.

Let $M' := M \cap (x = 0)$ be the submodule of rank two. For two elements $m_1, m_2 \in M'$ we write the line segment $\text{Conv}\{m_1, m_2\}$ simply as $[m_1, m_2]$.

Proposition 5 *Let $m_1, m_2, m_3 \in M'$ be three non-zero elements with $m_1 \neq m_2$. Set $A = [0, m_1], B = [0, m_2], E = [0, m_3]$. We assume that $\sharp A \cap M' = \sharp B \cap M' = 2$. If $m + 2E$ is not contained in the interior of the parallelogram $A + B$ for any $m \in M'$, then the lattice polytope $0 \times E + \text{Conv}\{0 \times A, 1 \times B\} \subset \mathbb{R} \times \mathbb{R}^2 \cong M_{\mathbb{R}}$ is normal.*

Proposition 6 *Let $m_1, m_2, m_3 \in M'$ be three non-zero elements. Set $A = [0, m_1], B = [0, m_2]$ and $E = [0, m_3]$. We assume that any two of A, B and E are not parallel. Then the lattice polytope $P = \text{Conv}\{0 \times (A + E), 1 \times (B + E)\} \subset M_{\mathbb{R}} = \mathbb{R} \times M'_{\mathbb{R}}$ is very ample.*

Definition Let P be a lattice polytope of vimension n. A vertex v of P is called *nonsingular* if the cone $C_v(P)$ is written as

$$\mathbb{R}_{\geq 0}u_1 + \cdots + \mathbb{R}_{\geq 0}u_n$$

such that $\{u, \ldots, u_n\}$ is a $\mathbb{Z}$-basis of M.

P is called *nonsingular* if all vertices are nonsingular. A face F of P is called *nonsingular* if F is nonsingular with respect to the sublattice $(\mathbb{R}F) \cap M$.

Remark 4 If P is a nonsingular lattice polytope, then it is very ample (see, for instance, Corollary 2.15 in [6]).

Proposition 7 *Let $m_1, m_2, m_3 \in M'$ be three non-zero elements with $m_1 \neq m_2$. Set $A = [0, m_1], B = [0, m_2], E = [0, m_3]$. We assume that $\sharp A \cap M' = \sharp B \cap M' = 2$. If there exists an $m \in M'$ such that $m + 2E$ is contained in the interior of the parallelogram $A + B$ and if one of two parallelograms $A + E$ and $B + E$ is nonsingular, then the lattice polytope $P = 0 \times E + \text{Conv}\{0 \times A, 1 \times B\} \subset M_{\mathbb{R}} = \mathbb{R} \times M'_{\mathbb{R}}$ is very ample but not normal.*

Remark 5 The nonsingularity condition of Proposition 7 is necessary. We give an example.

Set

$$A = [0, (1, 0)], \quad B = [0, (3, 10)], \quad E = [0, (a, b)]$$

and $P = 0 \times E + \text{Conv}\{0 \times A, 1 \times B\}$. If $(a, b) = (1, 2)$ or $(1, 4)$, then P is normal. On the other hand, if $(a, b) = (0, 1)$ or $(1, 3)$, then P is not normal.

Nonsingular Polytopes

Theorem 2 [8] *Let Q be a lattice polytope of dimension three and I a lattice interval such that the Minkowski sum $P = I + Q$ is a nonsingular lattice polytope of dimension n. Then P is normal.*

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