Boundary manifold and complement of complex line arrangement

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Proposition graph.

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A

in which we only keep the vertices of rank 1 and 0. It contain all the combi-

Let

A

graph of the positive MacLane ar-

The Figure below gives the incidence

Definition. The wiring diagram associated to the path \( \gamma \) is the subset of 

\[ W_A,\gamma = \{ (t, p^{-1}(\gamma(t)) \cap A ) \mid t \in [0,1] \} \]

For example, the wiring diagram of the positive MacLane arrangement is :

\[ \begin{array}{c}
1 & 2 & 3 \\
\gamma & \gamma & \gamma \\
\end{array} \]

The incidence graph is a subgraph of the Hasse diagram of the arrangement, 

in which we only keep the vertices of rank 1 and 0. It contain all the combi-

Definition. Let \( A \) be a line arrangement in \( \mathbb{C}P^2 \), and \( \Gamma(A) \) be the non-

oriented bipartite graph defined by :

\[\text{Point – vertices : } v_p, P \in \mathcal{P} \]
\[\text{Line – vertices : } v_L, L \in A \]

The edges of \( \Gamma(A) \) are of the form \( Y(L, P) \), with \( P \in \mathcal{P}, L \in A \) and \( P \in L \).

The Figure below gives the incidence graph of the positive MacLane arran-

The boundary manifold

The boundary manifold depends only on the combinatorics of \( A \). The fol-

following proposition describes a presentation of \( \pi_1(M(A)) \) from the incidence graph.

Proposition ([BGB12]). The fundamental group \( \pi_1(M(A)) \) admits the fol-

owing presentation:

- A set of generators \( \{ x_i \mid L_i \in A \} \), that represent the loops around the

lines.
- A set of generators \( \{ e_{i,j} \} \), indexed by the edges \( Y(L_i, L_j) \) that are not in the maximal tree.
- For each singular point \( P_s \), a set of relations given by the cyclic commutator \( [x_{i_1}x_{i_2}^{-1}, \ldots, x_{i_m}x_{i_{m+1}}^{-1}] \) where \( L_{i_1}, \ldots, L_{i_m} \) are the lines that pass through \( P_s \) and \( i_m \) is in uppermost.

Combatorics and wiring diagram

let \( \mathcal{P} = \{ P_1, \ldots, P_n \} \) be the set of the singular points of \( A \), \( p : \mathbb{C} \to C \) be 

a generic projection. We note \( \mathcal{Q} = \{ Q_1, \ldots, Q_k \} \) the images of the singular points of \( A \) by the projection \( p \).

Consider a path \( \gamma : [0,1] \to A \) with no self-intersection, and such that 

\( \mathcal{Q} \subset \gamma([0,1]) \).

Definition. The wiring diagram associated to the path \( \gamma \) is the subset of 

\[ W_A,\gamma = \{ (t, p^{-1}(\gamma(t)) \cap A ) \mid t \in [0,1] \} \]

The main result

Definition. For any cycle \( \gamma \) in \( W_A \), we define the upper word \( \sigma_\gamma \) by :

\[ \sigma_\gamma = \prod_{e \in E}(a_{e}(\gamma)) \]

where \( e(\sigma,\gamma) \) is 1 (resp. -1) if the crossing is positive (resp. negative), and 

\( a_e \) the word of Arvola of \( s \) and \( S \), the set of segment of \( W_A \) intersecting 

uppermost \( \gamma \).

In the example at the left, the segment \( s \) (dashed line) is the only one upper 

segment of the cycle \( \gamma \). The Arvola’s word of \( s \) is \( a_s = x_1x_2x_1^{-1} \). So we obtain that :

\[ \sigma_\gamma = x_2x_1^{-1}x_2^{-1} \]

because the crossing between \( \gamma \) and \( s \) is negative.

Definition. For any cycle \( \varepsilon \) in \( M(A) \), we define the uncrossing word \( \delta_\varepsilon \) as 

a product of the \( x_i \) such that \( \delta_\varepsilon \varepsilon \) is the path in \( \text{im}(\varepsilon) \) corresponding 

to \( \varepsilon \) (i.e. such that \( \forall \varepsilon \in \pi_1(\Gamma(A)), e \in \sigma^{-1}(\varepsilon) \Rightarrow \varepsilon(e) = \delta_\varepsilon(e) \)).

Let \( S \) be the normal sub-group of \( \pi_1(M(A)) \) generated by the elements 

\( \delta_\varepsilon \sigma^{-1} \), where \( \varepsilon \) are the cycles of \( \pi_1(M(A)) \).

Theorem ([BGB12]). Let \( A \) be a complex line arrangement, \( M_A \) be the 

boundary manifold, and \( \Gamma_A \) the incidence graph of \( A \). There exists a 
group \( S \) such that the following short sequence is exact :

\[ 0 \rightarrow S \xrightarrow{i_s} \pi_1(M(A)) \xrightarrow{\pi(0)} \pi_1(E(A)) \rightarrow 0, \]

where \( i_s \) is induced by the inclusion of \( M(A) \) in \( E(A) \).

Furthermore, a presentation of \( S \) can be computed from the wiring dia-

gram \( W_A,\gamma \).

Moreover, the generators of this presentation of \( S \) can be expressed in terms 

of the generators of Proposition below.

Sketch of the proof :

- The surjectivity of \( i_s \) comes from the Zariski-Van Kampen and the isomor-

phism between \( \pi_1(\mathbb{C} \setminus A) \) and \( \pi_1(\mathbb{C} \setminus (A - L_0)) \).
- Since \( S \) is constructed as a subgroup of \( \pi_1(M(A)) \), then \( \Phi \) is one-to-one.
- We glue on the cycle of the boundary manifold some 2-cells which give the 

retraction of \( \delta_\varepsilon \sigma \). So the composition map \( i_s \circ \Phi \) is zero.
- To show the exactness of the short exact sequence, we prove that the relation of 

the quotient \( \pi_1(M(A))/S \) implies the usual relation of \( \pi_1(E(A)) \).

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