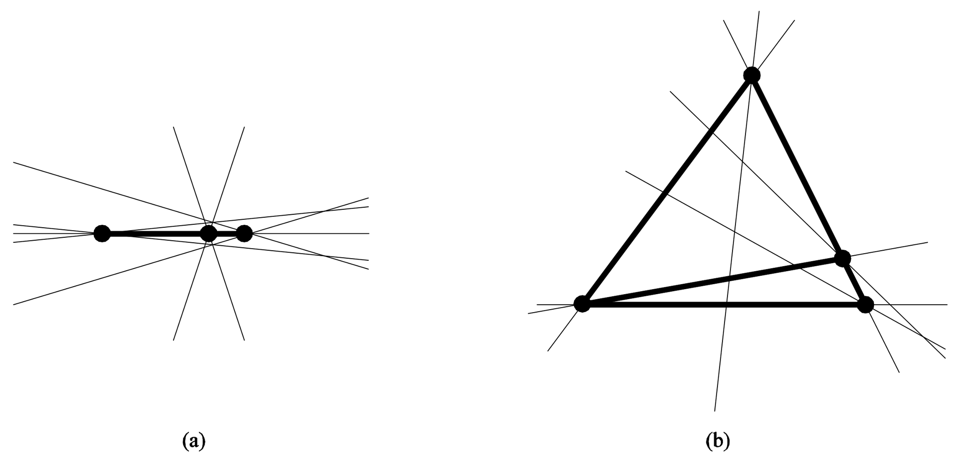
Fundamental Groups of Complements of Conic-Line Arrangements

Michael Friedman*, David Garber**

* Max Planck Institute for Mathematics, Bonn, Germany. mfriedman@mpim-bonn.mpg.de

** Department of Applied Mathematics, Holon Institute of Technology, Israel. garber@hit.ac.il

Definition [Fan]: Given a real line arrangement *L*, ***the graph G(L) of multiple points*** lies on the real part of *L*. It consists of the multiple points of *L* with the segments between the multiple points on lines which have at least two multiple points. If the arrangement contains 3 multiple points on the same line, then *G(L)* has 3 vertices on the same line.



Proposition [Fan, ELST]: A line arrangement L has a cycle-free graph G(L) $\Leftrightarrow \pi_1(\mathbb{C}^2 - L)$ is a direct sum of free groups and a free abelian group.

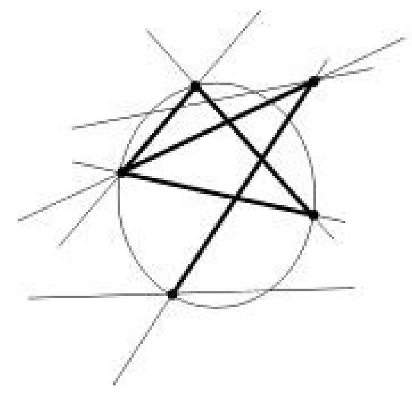
Extend the above definition to real Conic-Line arrangements:

Definition [FG]: (1) A ***real Conic-Line (CL) arrangement*** *A* is a collection of conics and lines in $\mathbb{C}^2$ where all the conics and the lines are defined over $\mathbb{R}$ and every singular point of the arrangement is in $\mathbb{R}^2$. In addition, for every conic C, $C \cap \mathbb{R}^2$ is not an empty set, neither a point nor a (double) line. Assume also that for every two component $\ell, \ell' \in A$, ℓ, ℓ' intersect transversally.

(2) The ***graph*** *G(A)* ***of the real CL arrangement*** *A*:

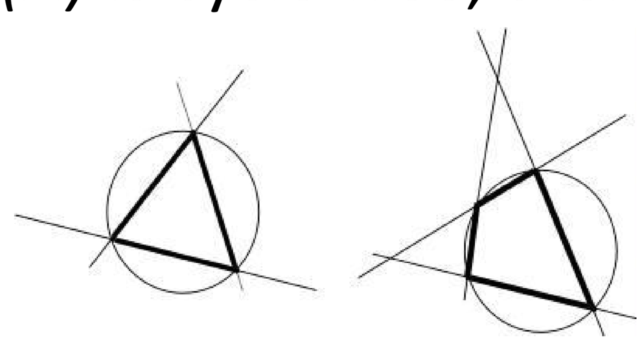
* Vertices: the multiple points.

* Edges: the segments on the lines connecting these points.



Proposition [FG]: Let *A* be a real CL arrangement with one conic. If the graph *G(A)* is cycle-free, then $\pi_1(\mathbb{C}^2 - A)$ is a direct sum of free groups and a free abelian group.

Remark: The inverse direction is not correct: For the arrangement A_3, $\pi_1(\mathbb{C}^2 - A_3)$ is abelian even though the graph is a cycle **[D]**.



The arrangements A_3, A_4

Definition [EGT, FG]: ***A conjugation-free geometric presentation of the fundamental group G of the complement of a real CL arrangement with k components:*** *G* has the following presentation:

Generators: $\{x_1, \ldots, x_m\}$, where *m=n* in the affine case, and *m=n-1* in the projective case.

Relations: $x_{i_k} x_{i_{k-1}} \cdots x_{i_1} = x_{i_{k-1}} \cdots x_{i_1} x_{i_k} = \cdots = x_{i_1} x_{i_k} \cdots x_{i_2}$ or $x_{i_1} = x_{i_2}$ for an increasing subsequence $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, \ldots, m\}$.

Proposition [FG]: Let *A* be a real CL arrangement with one conic.

(1) If the graph *G(A)* is cycle-free, then $\pi_1(\mathbb{C}^2 - A)$ has a conjugation-free geometric presentation.

(2) If the graph *G(A)* has one cycle, but the conic does not pass through all the multiple points, then $\pi_1(\mathbb{C}^2 - A)$ has a conjugation-free geometric presentation (and it is a direct sum of free groups and a free abelian group).

Question: What happens when the graph *G(A)* has one cycle and the conic pass through all the multiple points?

Example: Take the arrangement A_n: *n* lines that form a regular *n*-gon in $\mathbb{R}^2$ and a conic passing through all the edges of this *n*-gon (see A_3, A_4 above). Let $G_n = \pi_1(\mathbb{C}^2 - A_n)$.

Proposition [FG]: (1) G_3 is abelian (as was noted earlier), $G_4 = \mathbb{Z}^3 \oplus \mathbb{F}_2$ and both groups do **not** have a conjugation-free geometric presentation.

(2) G_{2k+1} for k≥1 is abelian, but does not have a conjugation-free geometric presentation.

(3) G_6 is not a direct sum of free groups and a free abelian group.

Question: What is the structure of G_{2k} for k≥3?

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