FREENESS OF HYPERPLANE ARRANGEMENTS AND DIVISORS

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Abstract. This is intended to deliver as a lecture note "Arrangements in Pyrénées". The main theme is free arrangements. We are discussing relations of freeness and splitting problems of vector bundles, freeness of Catalan and Shi arrangements, and K. Saito's theory of primitive derivations for Coxeter arrangements.

0. Introduction

Roughly speaking, there are two kind of objects in mathematics: general objects and specialized objects. In the study of general objects, individual objects are not so important, the totality of general objects is rather interesting (e.g. stable algebraic curves and moduli spaces). On the other hand, specialized objects are isolated, tend to be studied individually.

Let us fix a manifold (algebraic, complex analytic, what ever) X. Then the divisors on X are general objects. In 1970's Kyoji Saito [30] introduced the notion of free divisors with the motivation to compute Gauss-Manin connections for universal unfolding of isolated singularities. Free divisors are specialized objects. He found several basic results and almost all divisors are not free. He also did invariant theoretic study on Coxeter arrangements, and found several deep structures related to freeness. However, it has not been understood well (for me). At the almost same time, theory of free hyperplane arrangements was initiated by H. Terao.

In these lectures, we are concentrating around freeness of hyperplane arrangements. Let us pose three questions to which we are trying to answer in the series lectures.

1. How to prove freeness of arrangements?
2. Are there applications of free arrangements?
3. What is Saito's theory on Coxeter arrangements?

In early days, the freeness of arrangements was studied mainly from combinatorial view point. It was pointed out by Silvotti [38] and

Schenck [34, 25] that the freeness are equivalent to splitting of a reflexive sheaf on the projective space $\mathbb{P}^n$ into sum of line bundles ("splitting problem"). This view point has been a source of ideas of several recent studies on freeness of arrangements. We are trying to depict it in §1. These algebro-geometric study together with a result from §3, it is proved that (extended) Catalan and (extended) Shi arrangements are free. This fact has some applications to combinatorics of reflection arrangements of root systems (the Weyl arrangements) via Terao's factorization theorem [45] and Solomon-Terao's formula [40]. We are discussing these applications in §2. We also try to convince that a few open problems (including "Riemann Hypothesis" by Postnikov and Stanley [29]) seems to be related to the algebraic structures studied in §3. In §3, we sketch the invariant theoretic study of Coxeter arrangements by K. Saito. The notion "primitive derivation" plays the crucial role. Terao interpreted these structures in terms of derivation modules of Coxeter arrangements with multiplicities [47, 48, 52].

1. Splitting v.s. Freeness

In the first lecture, we are discussing freeness of divisors (especially hyperplane arrangements) and splitting problems of vector bundles. We are emphasizing parallelness and subtle differences.

1.1. **Splitting problems.** First let us recall the correspondence between graded modules and coherent sheaves on the projective scheme. (Basic reference is [22, Chap II §5]) Let $S = \mathbb{C}[x_1, x_2, \dots, x_\ell]$ be the polynomial ring and $\mathbb{P}^{\ell-1}_{\mathbb{C}} = \operatorname{Proj} S$ the projective $(\ell-1)$-space (denote $\mathbb{P}^{\ell-1}$ for simplicity). $\mathbb{P}^{\ell-1}$ is covered by open subsets U_{x_i} $(i = 1, \dots, \ell)$, where U_{x_i} is an open subset defined by $\{x_i \neq 0\}$. Let M be a graded S-module. Then M induces a sheaf $\widetilde{M}$ on $\mathbb{P}^{\ell-1}$, with sections

$$\Gamma(U_{x_i}, \widetilde{M}) = (M_{x_i})_0,$$

where $M_{x_i} = M \otimes_S S[\frac{1}{x_i}]$ is the localization by x_i and $(-)_d$ denotes the degree d component of the graded module. For $k \in \mathbb{Z}$, define the graded module $M(k)$ by shifting degrees by k, namely, $M(k)_d = M_{d+k}$. Denote $\mathcal{O} = \widetilde{S}$. The sheaf $\widetilde{S(k)}$ is a rank one module over $\mathcal{O}$, which is denoted by $\mathcal{O}(k)$.

Using the natural map $\Gamma(\mathbb{P}^{\ell-1}, \mathcal{E}) \times \Gamma(\mathbb{P}^{\ell-1}, \mathcal{F}) \longrightarrow \Gamma(\mathbb{P}^{\ell-1}, \mathcal{E} \otimes \mathcal{F})$, we can define a graded ring structure on $\Gamma_*(\mathcal{O}) := \bigoplus_{d\in\mathbb{Z}} \Gamma(\mathbb{P}^{\ell-1}, \mathcal{O}(d))$, which is isomorphic to S. More generally, for any sheaf ($\mathcal{O}$-module) $\mathcal{F}$ on $\mathbb{P}^{\ell-1}$,

$$\Gamma_*(\mathcal{F}) := \bigoplus_{d\in\mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{O}(d))$$

has a graded S-module structure. For a graded S-module M, $\Gamma_*(\widetilde{M})$ is expressed as

$$\begin{aligned}\Gamma_*(\widetilde{M}) &= \{(f_1, \dots, f_\ell) \mid f_i \in M_{x_i},\ f_i = f_j \text{ in } M_{x_i x_j}\} \\ &= \{f \in M_{x_1 x_2 \dots x_\ell} \mid \exists N \gg 0, x_i^N f \in M, \forall i = 1, \dots, \ell\}.\end{aligned}$$

Hence there is a natural homomorphism $\alpha : M \longrightarrow \Gamma_*(\widetilde{M})$. The above map α is not necessarily isomorphic, however, is isomorphic for graded modules appeared in this notes. We will see later.

Definition 1.1. A sheaf of $\mathcal{O}$-module $\mathcal{F}$ on $\mathbb{P}^n$ is said to be *splitting* if there exist integers $d_1, \dots, d_r \in \mathbb{Z}$ such that

$$\mathcal{F} \simeq \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r).$$

(Note that if we pose $d_1 \geq d_2 \geq \cdots \geq d_r$, the degrees are uniquely determined.)

Let $\mathcal{E}$ be an $\mathcal{O}$-module. Denote by $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O})$ the dual module of $\mathcal{E}$. $\mathcal{E}$ is called *reflexive* if the natural map $\mathcal{E} \longrightarrow \mathcal{E}^{\vee\vee}$ is isomorphic. $\mathcal{E}$ is called a *vector bundle* if it is locally free.

The torsion free $\mathcal{O}$-module on $\mathbb{P}^1$ is always splitting.

Theorem 1.2. *(Grothendieck's splitting theorem) Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^1$. Then $\mathcal{E}$ is splitting.*

A vector bundle $\mathcal{E}$ on $\mathbb{P}^n$, with $n \geq 2$ is not splitting in general. E.g., the tangent bundle $T_{\mathbb{P}^n}$ is irreducible rank n vector bundle on $\mathbb{P}^n$ for $n \geq 2$.

Let $\mathcal{E}$ be a torsion free sheaf. Let H be a hyperplane defined by a linear form α. Since $\alpha \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$, we have the following short exact sequence

$$0 \longrightarrow \mathcal{E}(-1) \xrightarrow{\alpha\cdot} \mathcal{E} \longrightarrow \mathcal{E}|_H \longrightarrow 0. \tag{1}$$

The short exact sequence (1) plays crucial role splitting problems.

Let $\mathcal{E}$ be a rank r vector bundle on $\mathbb{P}^n$. Then $\det \mathcal{E} := \bigwedge^r \mathcal{E}$ is a line bundle and called the *determinant bundle. The first Chern number* of $\mathcal{E}$ is the integer $c_1 \in \mathbb{Z}$ satisfying $\det \mathcal{E} = \mathcal{O}(c_1)$.

Proposition 1.3. *Let $\mathcal{E}$ be a rank r vector bundle on $\mathbb{P}^n$ with $n \geq 2$.*

(i) Let $\delta_i \in \Gamma(\mathcal{E} \otimes \mathcal{O}(-d_i))$ for certain $d_i \in \mathbb{Z}$, $i = 1, \dots, r$. Assume $\delta_1, \dots, \delta_r$ are linearly independent over rational function field (or equivalently, $\delta_1 \wedge \cdots \wedge \delta_r \in \Gamma(\det \mathcal{E} \otimes \mathcal{O}(-d_1 - \cdots - d_r))$ is nonzero) and $\sum_{i=1}^r d_i = c_1(\mathcal{E})$, then $\mathcal{E}$ is splitting and $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$.

(*ii*) *Let $H \subset \mathbb{P}^n$ be a hyperplane. If the restriction $\mathcal{E}|_H$ to H is splitting and the induced map*

$$\Gamma_*(\mathcal{E}) \longrightarrow \Gamma_*(\mathcal{E}|_H)$$

is surjective, then $\mathcal{E}$ is also splitting.

Proof. (*i*) Let $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$. Then $(\delta_1, \dots, \delta_r)$ determines a homomorphism

$$\mathcal{F} \longrightarrow \mathcal{E} \ : \ (f_1, \dots, f_r) \longmapsto f_1\delta_1 + \cdots + f_r\delta_r.$$

The Jacobian of this map is an element of $\Gamma(\mathcal{H}\mathrm{om}(\mathcal{O}(d_1+\cdots+d_r), \mathcal{O}(c_1))) = \Gamma(\mathcal{O}) = \mathbb{C}$. By assumption, the Jacobian is nowhere vanishing, hence $\mathcal{F} \simeq \mathcal{E}$.

(*ii*) Suppose that $\mathcal{E}|_H = \bigoplus_{i=1}^r F_i$ and $F_i \simeq \mathcal{O}_H(d_i)$. Then by surjectivity assumption, there is $\delta_i \in \mathcal{E} \otimes \mathcal{O}(-d_i)$ such that $\delta_i|_H$ is a nowhere vanishing section of $\Gamma(H, \mathcal{F}_i \otimes \mathcal{O}(-d_i)) = \mathbb{C}$. Since $\delta_1|_H, \dots, \delta_r|_H$ are linearly independent, so are $\delta_1, \dots, \delta_r$. Then by (*i*), $\mathcal{E}$ is also splitting. □

Remark 1.4. Comments to whom already familiar with free arrangements: (*i*) and (*ii*) in Proposition 1.3 are analogies of Saito's and Ziegler's criteria respectively. See Theorem 1.13 and Corollary 1.34.

Here we present some criteria for splitting.

Theorem 1.5. *Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}^n$.*

1. *(Horrocks) Assume $n \geq 2$. $\mathcal{F}$ is splitting $\Longleftrightarrow$ $H^i(\mathcal{F}(d)) = 0$, for any $0 < i < n$ and $d \in \mathbb{Z}$ $\Longleftrightarrow$ $H^1(\mathcal{F}(d)) = 0$, for any $d \in \mathbb{Z}$.*
2. *(Horrocks) Assume $n \geq 3$. Fix a hyperplane $H \subset \mathbb{P}^n$. Then $\mathcal{F}$ is splitting $\Longleftrightarrow$ $\mathcal{F}|_H$ is splitting.*
3. *(Bertone-Roggero, [15]) Assume $n \geq 2$. Fix a line $L \subset \mathbb{P}^n$ and assume that $\mathcal{E}|_L = \mathcal{O}_L(d_1) \oplus \dots \mathcal{O}_L(d_r)$. Then*

$$c_2(\mathcal{E}) \geq \sum_{i<j} d_i d_j,$$

and $\mathcal{E}$ is splitting if and only if the equality holds.

Proof. Here we give the proof for (1). The direction $\Longrightarrow$ is well-known ($H^i(\mathbb{P}^n, \mathcal{O}(d)) = 0$, for $0 < i < n$ and $d \in \mathbb{Z}$). Let us assume $H^1(\mathbb{P}^n, \mathcal{F}(d)) = 0$ for $d \in \mathbb{Z}$. Let us first consider the case $n = 2$. By Grothendieck's splitting theorem, $\mathcal{F}|_H$ is splitting. By the long exact sequence associated with (1), we have the surjectivity of $\Gamma_*(\mathcal{F}) \longrightarrow \Gamma_*(\mathcal{F}|_H)$. Hence by Proposition 1.3 (*i*), $\mathcal{F}$ is splitting. For $n \geq 3$, it is proved by induction. □

Remark 1.6. Horrocks' restriction criterion (2) is generalized to reflexive sheaves ([9]).

We will give a refinement of (3) for $n = r = 2$.

1.2. **Basics of arrangements.** Let V be an ℓ-dimensional vector space. A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \dots, H_n\}$ is called a *hyperplane arrangement*. For each hyperplane H_i we fix a defining equation α_i such that $H_i = \alpha_i^{-1}(0)$. An arrangement $\mathcal{A}$ is called *central* if each H_i passes the origin $0 \in V$. In this case, the defining equation $\alpha_i \in V^*$ is linear homogeneous. Let $L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. Define a partial order on $L(\mathcal{A})$ by $X \leq Y \iff Y \subseteq X$ for $X, Y \in L(\mathcal{A})$. Note that this is reverse inclusion.

Define a *rank function* on $L(\mathcal{A})$ by $r(X) = \operatorname{codim} X$. Write $L^p(\mathcal{A}) = \{X \in L(\mathcal{A}) |\ r(X) = p\}$. We call $\mathcal{A}$ *essential* if $L^\ell(\mathcal{A}) \neq \emptyset$.

Let $\mu : L(\mathcal{A}) \to \mathbb{Z}$ be the *Möbius function* of $L(\mathcal{A})$ defined by

$$\mu(X) = \begin{cases} 1 & \text{for } X = V \\ -\sum_{Y<X} \mu(Y), & \text{for } X > V. \end{cases}$$

The *characteristic polynomial* of $\mathcal{A}$ is $\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}$. The characteristic polynomial is characterized by the following recursive relations.

Proposition 1.7. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in V. Let $\mathcal{A}' = \{H_1, \dots, H_{n-1}\}$ and $\mathcal{A}'' = H_n \cap \mathcal{A}'$ the induced arrangement on H_n. Then*

- *in case $\mathcal{A}$ is empty, $\chi(\emptyset, t) = t^{\dim V}$, and*
- *$\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t)$.*

We also define numbers $b_i(\mathcal{A})$ $(i = 1, \dots, \ell)$ by the formula

$$\chi(\mathcal{A}, t) = \sum_{i=0}^{\ell} (-1)^i b_i(\mathcal{A}) t^{\ell - i}.$$

This naming and the importance of the characteristic polynomial in combinatorics would be justified by the following result.

Theorem 1.8. (1) *If $\mathcal{A}$ is an arrangement in $\mathbb{F}_q^\ell$ (vector space over a finite field $\mathbb{F}_q$), then $|\mathbb{F}_q^\ell \setminus \bigcup_{H \in \mathcal{A}} H| = \chi(\mathcal{A}, q)$.*

(2) *If $\mathcal{A}$ is an arrangement in $\mathbb{C}^\ell$, then the i-th Betti number of the complement is $b_i(\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H) = b_i(\mathcal{A})$.*

(3) *If $\mathcal{A}$ is an arrangement in $\mathbb{R}^\ell$, then $|\chi(\mathcal{A}, -1)|$ is the number of chambers and $|\chi(\mathcal{A}, 1)|$ is the number of bounded chambers.*

(1) of the above theorem can be used for the computation of of $\chi(\mathcal{A}, t)$ for $\mathcal{A}$ defined over $\mathbb{Q}$ (or $\mathbb{A}$). It is sometimes called "Finite field method". Athanasiadis pointed out that we may drop the assumption

"field". Let $\mathcal{A}$ be a hyperplane arrangement such that each $H \in \mathcal{A}$ is defined by a linear form α_H with $\mathbb{Z}$-coefficient.

Theorem 1.9. *Let $\mathcal{A}$ be a hyperplane arrangement such that each $H \in \mathcal{A}$ is defined by a linear form α_H of $\mathbb{Z}$-coefficients. For an positive integer $m > 0$, consider $\overline{H} = \{x \in (\mathbb{Z}/m\mathbb{Z})^\ell \mid \alpha_H(x) \equiv 0 \mod m\}$. There exists a positive integer N which is depending only on $\mathcal{A}$ such that if $m > N$ and coprime to N, then*

$$|(\mathbb{Z}/m\mathbb{Z})^n \setminus \bigcup_{H\in\mathcal{A}} \overline{H}| = \chi(\mathcal{A}, m).$$

Athanasiadis systematically used this result to compute characteristic polynomials. I recommend [13, 14].

1.3. **Basics of free arrangements.** Let $V = \mathbb{C}^\ell$ be a complex vector space with coordinate $(x_1, \cdots, x_\ell)$, $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central hyperplane arrangement, namely, assume $0 \in H_i$ for all $i = 1, \dots, n$. We denote by $\mathrm{Der}_V = \bigoplus_{i=1}^{\ell} S\frac{\partial}{\partial x_i}$ the set of polynomial vector fields on V (or S-derivations) and by $\Omega_V^p = \bigoplus_{i_1<\cdots<i_p} S dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ the set of polynomial differential p-forms.

Definition 1.10. *Let $\theta = \sum_{i=1}^{\ell} f_i \partial_{x_i}$ be a polynomial vector field. θi said to be homogeneous of polynomial degree d when $f_1, \dots, f_\ell$ are homogeneous polynomial of degree d. It is denoted by $\mathrm{pdeg}\,\theta = d$.*

Usually the degree of θ is considered to be $\mathrm{pdeg}\,\theta - 1$.

Let us denote by $S = \mathbb{C}[x_1, \dots, x_\ell]$ the polynomial ring and fix $\alpha_i \in V^*$ a defining equation of H_i, i.e., $H_i = \alpha_i^{-1}(0)$.

Definition 1.11. *A multiarrangement is a pair $(\mathcal{A}, \mathbf{m})$ of an arrangement $\mathcal{A}$ with a map $\mathbf{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, called the multiplicity.*

An arrangement $\mathcal{A}$ can be identified with a multiarrangement with constant multiplicity $m \equiv 1$, which is sometimes called a *simple arrangement.* With this notation, the main object is the following module of S-derivations which has contact to each hyperplane of order m. We also put $Q = Q(\mathcal{A}, \mathbf{m}) = \prod_{i=1}^{n} \alpha_i^{\mathbf{m}(H_i)}$ and $|\mathbf{m}| = \sum_{H\in\mathcal{A}} \mathbf{m}(H)$.

Definition 1.12. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement, and define the module of vector fields logarithmic tangent to $\mathcal{A}$ with multiplicity $\mathbf{m}$ (logarithmic vector field) by*

$$D(\mathcal{A}, \mathbf{m}) = \{\delta \in \mathrm{Der}_S \,|\delta\alpha_i \in (\alpha_i)^{\mathbf{m}(H_i)}, \forall i\},$$

and and differential forms with logarithmic poles along $\mathcal{A}$ (logarithmic forms) by

$$\Omega^p(\mathcal{A},\mathbf{m}) = \left\{ \omega \in \frac{1}{Q}\Omega^p_V \,\middle|\, d\alpha_i \wedge \omega \text{ does not have pole along } H_i, \forall i \right\}.$$

The module $D(\mathcal{A},\mathbf{m})$ is obviously a graded S-module. It is proved in [30] that $D(\mathcal{A},\mathbf{m})$ and $\Omega^1(\mathcal{A},\mathbf{m})$ are dual modules to each other. Therefore, they are reflexive modules. A multiarrangement $(\mathcal{A},\mathbf{m})$ is said to be *free* with exponents $(e_1,\dots,e_\ell)$ if and only if $D(\mathcal{A},\mathbf{m})$ is an S-free module and there exists a basis $\delta_1,\dots,\delta_\ell \in D(\mathcal{A},\mathbf{m})$ such that $\det\delta_i = e_i$. Here note that the degree $\deg\delta$ of a derivation δ is the polynomial degree, in other words, $\deg(\delta f) = \deg\delta + \deg f - 1$ for a homogeneous polynomial f. When $\mathbf{m} \equiv 1$, $D(\mathcal{A},1)$ and $\Omega^p(\mathcal{A},1)$ is denoted by $D(\mathcal{A})$ and $\Omega^p(\mathcal{A})$ for simplicity. An arrangement $\mathcal{A}$ is said to be free if $(\mathcal{A},1)$ is free. The Euler vector field $\theta_E = \sum_{i=1}^{\ell} x_i\partial_i$ is always contained in $D(\mathcal{A})$ for simple case.

Let $\delta_1,\dots,\delta_\ell \in D(\mathcal{A},\mathbf{m})$. Then $\delta_1\wedge\cdots\wedge\delta_\ell$ is divisible by $Q(\mathcal{A},\mathbf{m})\frac{\partial}{\partial x_1}\wedge\cdots\wedge\frac{\partial}{\partial x_\ell}$. The determinant of coefficient matrix of $\delta_1,\dots,\delta_\ell$ can be used for characterize freeness.

Theorem 1.13. *(Saito's criterion,* [30]*) Let $\delta_1,\dots,\delta_\ell \in D(\mathcal{A},\mathbf{m})$. Then the following are equivalent:*

(i) *$D(\mathcal{A},\mathbf{m})$ is free with basis $\delta_1,\dots,\delta_\ell$, i. e., $D(\mathcal{A},\mathbf{m}) = S\cdot\delta_1 \oplus \cdots \oplus S\cdot\delta_\ell$.*
(ii) *$\delta_1\wedge\cdots\wedge\delta_\ell = c\cdot Q(\mathcal{A},\mathbf{m})\cdot\frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_\ell}$, where $c \in \mathbb{C}^*$.*
(iii) *$\delta_1,\dots,\delta_\ell$ are linearly independent over S and $\sum_{i-1}^{\ell}\deg\delta_i = |\mathbf{m}| = \sum_{H\in\mathcal{A}}\mathbf{m}(H)$.*

From Saito's criterion, we also obtain that if a multiarrangement $(\mathcal{A},\mathbf{m})$ is free with exponents $(e_1,\dots,e_\ell)$, then $|\mathbf{m}| = \sum_{i=1}^{\ell} e_i$.

Proposition 1.14. *$\mathcal{A}$ is free, then $\mathcal{A}$ is locally free, i. e., $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}$ is free for any $X \in L(\mathcal{A})$.*

For simple arrangement case, there is a good connection of these modules with the characteristic polynomial. The following theorem shows that $D(\mathcal{A})$ determines the characteristic polynomial.

The following result shows that $D(\mathcal{A})$ determines the characteristic polynomial $\chi(\mathcal{A},t)$.

Theorem 1.15. *(Solomon-Terao's formula* [40]*) Denote by* $H(\Omega^p(\mathcal{A}), x) \in \mathbb{Z}[[x]][x^{-1}]$ *the Hilbert series of the graded module* $\Omega^p(\mathcal{A})$*. Define*

$$\Phi(\mathcal{A}; x, y) = \sum_{p=0}^{\ell} H(\Omega^p(\mathcal{A}), x) y^p. \tag{2}$$

Then

$$\chi(\mathcal{A}, t) = \lim_{x \to 1} \Phi(\mathcal{A}; x, t(1-x) - 1). \tag{3}$$

In particular, for free arrangements, we have the following beautiful formula, which is known as Terao's factorization theorem.

Corollary 1.16. *(*[45]*) Suppose that* $\mathcal{A}$ *is a free arrangement with exponents* $(e_1, \dots, e_\ell)$*. Then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - e_i). \tag{4}$$

Remark 1.17. *There is a notion of the characteristic polynomial of a multiarrangement* $(\mathcal{A}, \mathbf{m})$ [6]*. However it can not be defined combinatorially, rather by the Solomon-Terao's formula for* $\Omega^p(\mathcal{A}, \mathbf{m})$*.*

Example 1.18. (Braid arrangement, A_{n-1}-type arrangement) Let $H_{ij} = \{(x_1, \dots, x_\ell)\mathbb{C}^\ell \mid x_i = x_j\}$. Consider the arrangement $\mathcal{A} = \{H_{ij} \mid 1 \le i < j \le n\}$. In other word $Q(\mathcal{A}) = \prod_{i<j}(x_i - x_j)$.

The characteristic polynomial of this arrangement is easily computed by finite field method. For the complement with $\otimes \mathbb{F}_q$ is expressed as

$$\{(x_1, \dots, x_n) \in \mathbb{F}_q^n \mid x_i \neq x_j, \text{ for } i \neq j\}$$

It is naturally bijective to (ordered) choices of n distinct elements from $\mathbb{F}_q$. Hence the cardinality is

$$|\mathbb{F}_q^n \setminus \bigcup_{i<j} H_{ij}| = q(q-1)\dots(q-n+1),$$

we have $\chi(\mathcal{A}, t) = t(t-1)(t-2)\dots(t-n+1)$.

Furthermore, $\mathcal{A}$ is a free arrangement. Indeed putting

$$\begin{aligned}
\delta_0 &= \partial_{x_1} + \partial_{x_2} + \cdots + \partial_{x_n}, \\
\delta_1 &= x_1\partial_{x_1} + x_2\partial_{x_2} + \cdots + x_n\partial_{x_n}, \\
\delta_2 &= x_1^2\partial_{x_1} + x_2^2\partial_{x_2} + \cdots + x_n^2\partial_{x_n}, \\
&\dots \\
\delta_{n-1} &= x_1^{n-1}\partial_{x_1} + x_2^{n-1}\partial_{x_2} + \cdots + x_n^{n-1}\partial_{x_n}.
\end{aligned}$$

Then $\delta_k(x_i - x_j) = x_i^k - x_j^k$, which is divisible by $(x_i - x_j)$. Hence $\delta_k \in D(\mathcal{A})$. Furthermore, by Vandermonde's formula

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix} = \pm \prod_{1 \le i < j \le n} (x_i - x_j),$$

and by Saito's criterion, we may conclude $\delta_0, \dots, \delta_{n-1}$ is a basis of $D(\mathcal{A})$. Hence $\mathcal{A}$ is-free with exponents $(0, 1, \dots, n-1)$.

To conclude this section, we note that the module of logarithmic vector fields is recovered from the sheafification:

$$D(\mathcal{A}, \mathbf{m}) \xrightarrow{\simeq} \Gamma_*(\widetilde{D(\mathcal{A}, \mathbf{m})}). \tag{5}$$

Therefore freeness of $(\mathcal{A}, \mathbf{m})$ is equivalent to the splitting of $D(\mathcal{A}, \mathbf{m})$.

Proposition 1.19. *$(\mathcal{A}, \mathbf{m})$ is free with exponents $(d_1, \dots, d_\ell)$ if and only if $\widetilde{D(\mathcal{A}, \mathbf{m})} \simeq \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \oplus \cdots \oplus \mathcal{O}(-d_\ell)$.*

1.4. **2-multiarrangements.** To emphasize the difficulty and recent developments on 2-multiarrangements, we devote a subsection to present results on 2-multiarrangements.

Simple arrangements $\mathcal{A} = \{H_1, \dots, H_n\}$ in dimension two is always free with exponents $(1, n-1)$. We can construct an example of basis explicitly as follows:

$$\delta_1 = x\partial_x + y\partial_y, \ \delta_2 = (\partial_y Q)\partial_x - (\partial_x Q)\partial_y.$$

The multiarrangement $(\mathcal{A}, \mathbf{m})$ in dimension two is also free. There are two ways to prove it. First idea is based on $D(\mathcal{A}, \mathbf{m})$ is a reflexive module. Then 2-dimension + reflexivity implies free. Another is based on the isomorphism

$$D(\mathcal{A}, \mathbf{m}) \xrightarrow{\simeq} \Gamma_*(\widetilde{D(\mathcal{A}, \mathbf{m})}).$$

If $\mathcal{A}$ is in dimension two, the sheafification $\widetilde{D(\mathcal{A}, \mathbf{m})}$ is a torsion free sheaf on $\mathbb{P}^1$. By Grothendieck splitting theorem, we conclude $D(\mathcal{A}, \mathbf{m})$ is free. We have the following.

Proposition 1.20. *Let $(\mathcal{A}, \mathbf{m})$ be a 2-multiarrangement. Then it is free and the exponents (d_1, d_2) satisfies $d_1 + d_2 = |\mathbf{m}|$.*

However:

- The determination of exponents are highly difficult.

- Exponents of 2-multiarrangements are related to the freeness of 3-simple arrangements. (See §1.5)

The following lemma is useful for decision of the exponents.

Lemma 1.21. *Let* $(\mathcal{A}, \mathbf{m})$ *be a 2-multiarrangement. Assume that* $\delta \in D(\mathcal{A}, \mathbf{m})$*, the coefficient degree satisfies* $d = \operatorname{pdeg} \delta \leq \frac{|\mathbf{m}|}{2}$ *and no divisor of* δ *is contained in* $D(\mathcal{A}, \mathbf{m})$*. Then* $\exp(\mathcal{A}, \mathbf{m}) = (d, |\mathbf{m}| - d)$*.*

Proof. Suppose that $\exp(\mathcal{A}, \mathbf{m}) = (d_1, d_2)$ with $d_1 \leq d_2$. Then clearly $d_1 \leq d$. There exists δ_1 of $\operatorname{pdeg} \delta_1 = d_1$. Since $d \leq \frac{|\mathbf{m}|}{2} = \frac{d_1+d_2}{2}$, we have $d_1 \leq d \leq d_2$. If $d_1 < d$, then we have $d_1 < d < d_2$. Hence δ can be expressed as $\delta = F \cdot \delta_1$ with some polynomial F of $\deg F > 0$. But this contradicts to the assumption that no nontrivial divisor of δ is contained in $D(\mathcal{A}, \mathbf{m})$. So $\deg F = 0$ and we have $d_1 = d, d_2 = |\mathbf{m}| - d$. □

Let us begin with typical cases.

Proposition 1.22. Let $(\mathcal{A}, \mathbf{m})$ be a 2-multiarrangement, we may assume that $m_i = \mathbf{m}(H_i)$ satisfies $m_1 \geq m_2 \geq \cdots \geq m_n > 0$. Set $m = \sum_{i=1}^n m_i$.

(i) If $m_1 \geq \frac{m}{2}$, then the exponents are $\exp(\mathcal{A}, \mathbf{m}) = (m_1, m - m_1)$.
(ii) if $n \geq \frac{m}{2} + 1$, then $\exp(\mathcal{A}, \mathbf{m}) = (m - n + 1, n - 1)$.
(iii) If $m_1 = m_2 = \cdots = m_n = 2$, then $\exp(\mathcal{A}, \mathbf{m}) = (n, n)$.
(iv) When $n = 3$, if $m_1 \leq m_2 + m_3$, then

$$\exp(\mathcal{A}, \mathbf{m}) = \begin{cases} (k, k), & \text{if } |\mathbf{m}| = 2k, \\ (k, k+1), & \text{if } |\mathbf{m}| = 2k + 1. \end{cases}$$

Proof. (i) We can set the coordinate (x, y) such that $H_1 = \{x = 0\}$, in other words, $\alpha_1 = x$. Set $\delta = (\prod_{i=2}^n \alpha^{m_i}) \cdot \partial_y$. Then $\delta x = 0$ and $\delta \alpha_i \in (\alpha_i)^{m_i}$ for $i \geq 2$. Hence $\delta \in D(\mathcal{A}, \mathbf{m})$. We also have

$$\operatorname{pdeg} \delta = m_2 + \dots m_n = |\mathbf{m}| - m_1 \leq \frac{|\mathbf{m}|}{2},$$

and no nontrivial devisor of δ is not contained in $D(\mathcal{A}, \mathbf{m})$. From Lemma 1.21, $\exp(\mathcal{A}, \mathbf{m}) = (m_1, |\mathbf{m}| - m_1)$.

(ii) Let us define δ as

$$\delta = \frac{\prod_{i=1}^n \alpha_i^{m_i}}{\prod_{i=1}^n \alpha_i} \cdot \theta_E,$$

where $\theta_E = x\partial_x + y\partial_y$ is the Euler vector field. Then since $\theta\alpha = \alpha$ for any linear form α, $\delta \in D(\mathcal{A}, \mathbf{m})$. From the assumption, we have

$$\operatorname{pdeg} \delta = |\mathbf{m}| - n + 1 \leq n - 1.$$

Since $(|\mathbf{m}|-n+1)+(n-1) = |\mathbf{m}|$, we have $|\mathbf{m}|-n+1 = \operatorname{pdeg}\delta \leq \frac{|\mathbf{m}|}{2}$. It is also easily checked that δ does not have non trivial divisor which is contained in $D(\mathcal{A}, \mathbf{m})$. Hence we have $\exp(\mathcal{A}, \mathbf{m}) = (|\mathbf{m}|-n+1, n-1)$.

(iii) and (iv) are proved by explicit constructions of basis. See [50, Exapmple 2.2] and [49] respectively. (Both are highly nontrivial.) □

Thus if either $\max\{\mathbf{m}(H) \mid H \in \mathcal{A}\}$ is large (not less than the half of $|\mathbf{m}| = \sum \mathbf{m}(H)$) or the number of lines $n = |\mathcal{A}|$ is large (not less than $\frac{|\mathbf{m}|}{2}+1$), then the exponents are combinatorially determined. This motivate us to the following definition.

Definition 1.23. *The multiplicith* $\mathbf{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ *is said to be* balanced *if* $\mathbf{m}(H) \leq \frac{\sum_{H\in\mathcal{A}} \mathbf{m}(H)}{2}$ *for all* $H \in \mathcal{A}$.

As we have seen, if the multiplicity is balanced, the exponents are combinatorially determined. However the exponents are not combinatorially determined in general balanced case.

Example 1.24. *Let* $(\mathcal{A}_t, \mathbf{m})$ *be a multiarrangement defined by*

$$Q(\mathcal{A}_t, \mathbf{m}) = x^3y^3(x+y)^1(tx-y)^1,$$

where $t \in \mathbb{C} \setminus \{0, 1, -1\}$. *Then exponents is*

$$\exp(\mathcal{A}_t, \mathbf{m}) = \begin{cases} (3,5), & \text{if } t = 1, \\ (4,4), & \text{if } t \neq 1. \end{cases}$$

Indeed, it is easily seen that

$$\delta_1 = x^3\partial_x + y^3\partial_y,$$
$$\delta_2 = x^5\partial_x + y^5\partial_y,$$

form a basis of $D(\mathcal{A}_1, \mathbf{m})$. *For* $n \neq 1$ *(and* $t \neq 0, -1$*),* $\delta_1 \notin D(\mathcal{A}_t, \mathbf{m})$. *But* $(tx-y)\delta_1 \in D(\mathcal{A}_t, \mathbf{m})$ *with* $\operatorname{pdeg} = 4$. *If there exists an element of* $D(\mathcal{A}_t, \mathbf{m})$ *of* $\operatorname{pdeg} = 3$, *it should be a divisor of* $(tx-y)\delta_1$. *It is impossible. Thus exponents for other cases is* $(4,4)$.

We may observe that any 4-lines can be moved to (by $PGL_2(\mathbb{C})$-action) to $xy(x+y)(tx-y)$ with $t \in \mathbb{C} \setminus \{0, -1\}$. On a Zarisky open subset of the parameter space $\mathbb{C} \setminus \{0, 1, -1\} \subset \mathbb{C} \setminus \{0, -1\}$, the exponents is $(4,4)$ and at $t = 1$, it becomes $(3,5)$. This generally happens. We shall prove the upper-semicontinuity on the parameter spamce of the following function.

Definition 1.25. *Put* $\exp(\mathcal{A}, \mathbf{m}) = (d_1, d_2)$. *Then we denote*

$$\Delta(\mathcal{A}, \mathbf{m}) = |d_1 - d_2|.$$

The difference of exponents $\Delta(\mathcal{A},\mathbf{m})$ is a function on $\mathcal{A}$ and $\mathbf{m}$. We first fix $\mathbf{m}$. The parameter space of $\mathcal{A}$ can be described as

$$\mathcal{M}_n = \{(H_1,\dots,H_n) \in (\mathbb{P}^{1*})^n \mid H_i \neq H_j, \ \ \text{for } i \neq j\}$$

Proposition 1.26. *Fix the multiplicity* $\mathbf{m} : \{1,\dots,n\} \to \mathbb{Z}_{>0}$. *Then*

$$\Delta : \mathcal{M}_n \longrightarrow \mathbb{Z}_{>0}, \ (\mathcal{A} \longmapsto \Delta(\mathcal{A},\mathbf{m}))$$

is upper semicontinuous, i. e., $\{\Delta < k\} \subset \mathcal{M}_n$ *is a Zarisky open subset for any* $k \in \mathbb{R}$.

Proof. It suffices to prove that $\{\Delta \geq k\}$ is a Zarisky closed in $\mathcal{M}_n$. Since $d_1 + d_2 = |\mathbf{m}|$, $\Delta(\mathcal{A},\mathbf{m}) \geq k$ if and only if there exists $\delta \in D(\mathcal{A},\mathbf{m})$ such that $\operatorname{pdeg}\delta \leq \lfloor \frac{|\mathbf{m}|}{2} - \frac{k}{2} \rfloor$. Thus we consider when $\delta \in D(\mathcal{A},\mathbf{m})$ of $\operatorname{pdeg}\delta = \lfloor \frac{|\mathbf{m}|}{2} - \frac{k}{2} \rfloor$ exists. Put $d = \lfloor \frac{|\mathbf{m}|}{2} - \frac{k}{2} \rfloor$, $\alpha_i = p_i x + q_i y$ and

$$\delta = (a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d)\partial_x + (b_0x^d + b_1x^{d-1}y + \cdots + b_dy^d)\partial_y.$$

The assertion $\delta\alpha_i \in (\alpha_i^{m_i})$ is equivalent to

(6) $\delta\alpha_i = (p_ix + q_iy)^{m_i}(c_0x^{d-m_i} + c_1x^{d-m_i-1}y + \dots c_{d-m_i}y^{d-m_i}).$

Hence the existence of $\delta \in D(\mathcal{A},\mathbf{m})$ of degree d is equivalent to the existence of the solution to the system of linear equations (6) on a_i, b_i and c_i. It is a Zarisky closed condition on the parameters p_i and q_i. □

The following is the two fundamental results on exponents of 2-multiarrangements.

Theorem 1.27. *Let* $\mathbf{m} : \{1,\dots,n\} \longrightarrow \mathbb{Z}_{>0}$ *be a balanced multiplicity and* $\mathcal{A} = \{H_1,\dots,H_n\}$ *is a 2-arrangement.*

(i) *(Wakefield-Yuzvinsky* [50]*) For generic* $\mathcal{A}$, $\Delta(\mathcal{A},\mathbf{m}) \leq 1$.

(ii) *(Abe* [2]*)* $\Delta(\mathcal{A},\mathbf{m}) \leq n-2$.

The proof of (i) is a careful extension of that of upper semicontinuity (Proposition 1.26). See papers cited for proof. The proof of (ii) is in very different nature. Abe ([2] and Abe-Numata [3]) first fix $\mathcal{A}$ and consider Δ as a function from the set of multiplicities $\mathbb{Z}^n_{>0}$ to $\mathbb{Z}_{\geq 0}$,

$$\Delta : \mathbb{Z}^n_{>0} \longrightarrow \mathbb{Z}_{\geq 0}, \ \mathbf{m} \longmapsto \Delta(\mathcal{A},\mathbf{m}).$$

They studied the structure of this function in great detail. The proof of (ii) is based on this.

(i) tells the generic behavior of the function Δ. (ii) tells the upper bound of Δ for balanced multiplicities. As far as I know, the examples $(\mathcal{A},\mathbf{m})$ attaining the upper bound of Δ is related to interesting free arrangements. Abe proved Terao's conjecture for related class of 3-arrangements [2] (Example 1.41).

Problem 1.28. *Give a unified proof for Theorem 1.27 (i) and (ii).*

1.5. **Multiarrangements and free arrangements.** Multiarrangements appear as restrictions of simple arrangements. Namely, let $\mathcal{A}$ be a arrangement in V of $\dim V = \ell$. For $H \in \mathcal{A}$ let us denote by $\mathcal{A}^H$ the induced arrangement on H.

Definition 1.29. *Define the function* $\mathbf{m}^H : \mathcal{A}^H \longrightarrow \mathbb{Z}_{>0}$ *by*

$$X \in \mathcal{A}^H \longmapsto \sharp\{H' \in \mathcal{A} \mid H' \supset X\} - 1.$$

We call $(\mathcal{A}^H, \mathbf{m}^H)$ *the Ziegler's multirestriction.*

Example 1.30. *Let* $V = \mathbb{C}^3$ *with coordinates* x, y, z. *Put* $H_1 = \{z = 0\}, H_2 = \{x = 0\}, H_3 = \{y = 0\}, H_4 = \{x - z = 0\}, H_5 = \{x + z = 0\}, H_6 = \{y - z = 0\}, H_7 = \{y + z = 0\}, H_8 = \{x - y = 0\}, H_9 = \{x + y = 0\}$. *Then* $\mathcal{A} = \{H_1, \ldots, H_9\}$ *is free with exponents* $(1, 3, 5)$. *Ziegler's multirestriction to* $(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$ *is* $x^3y^3(x - y)(x + y)$. *(See Figure 1)*

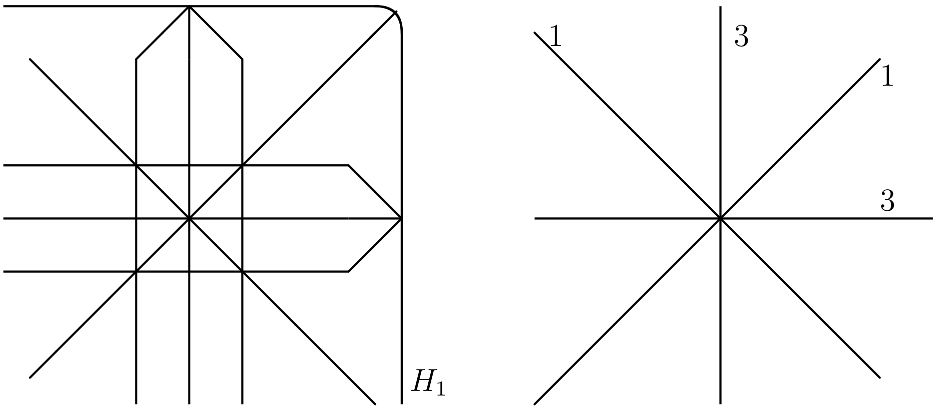

FIGURE 1. $\mathcal{A} = \{H_1, \ldots, H_9\}$ and $(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$

Definition 1.31. *Fix a hyperplane* $H_1 \in \mathcal{A}$. *Then we define a submodule* $D_1(\mathcal{A})$ *of* $D(\mathcal{A})$ *by*

$$D_1(\mathcal{A}) = \{\delta \in D(\mathcal{A}) \mid \delta\alpha_{H_1} = 0\}.$$

Lemma 1.32. *Under notations as above,* $D(\mathcal{A}) = S \cdot \theta_E \oplus D_1(\mathcal{A})$.

Proof. Let $\delta \in D(\mathcal{A})$. Since $\delta - \frac{\delta\alpha_{H_1}}{\alpha_{H_1}} \cdot \theta_E$ is in $D_1(\mathcal{A})$, $\delta = \frac{\delta\alpha_{H_1}}{\alpha_{H_1}} \cdot \theta_E + \left(\delta - \frac{\delta\alpha_{H_1}}{\alpha_{H_1}} \cdot \theta_E\right)$ gives a desired decomposition. □

Theorem 1.33. *(Ziegler* [59]*) Notations as above.*

(i) *If* $\delta \in D_1(\mathcal{A})$, *then* $\delta|_{H_1} \in D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$.

(ii) *If $\mathcal{A}$ is free with exponents $(1, d_2, \dots, d_\ell)$, then $(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$ is free with exponents $(d_2, \dots, d_\ell)$.*

Proof. We can choose coordinates $x_1, \dots, x_\ell$ in such a way that $x_1 = \alpha_{H_1}$. Let $X \in \mathcal{A}^{H_1}$ and put

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supset X\} = \{H_1, H_{i_1}, H_{i_2}, \dots, H_{i_m}\}.$$

Since $H \cap H_{i_1} = \cdots = H \cap H_{i_m} = X$, $\alpha_{i_p}|_{x_1=0}$ determines the same hyperplane. Thus we may assume that α_{i_p} has the form

$$\begin{aligned}\alpha_{i_1}(x_1, \dots x_\ell) &= c_1 x_1 + \alpha'(x_2, \dots, x_\ell)\\ \alpha_{i_2}(x_1, \dots x_\ell) &= c_2 x_1 + \alpha'(x_2, \dots, x_\ell)\\ &\cdots\cdots\\ \alpha_{i_m}(x_1, \dots x_\ell) &= c_m x_1 + \alpha'(x_2, \dots, x_\ell),\end{aligned}$$

where $c_1, \dots, c_m$ are mutually distinct. Let $\delta \in D_1(\mathcal{A})$. By definition,

$$\delta(c_k x_1 + \alpha'(x_2, \dots, x_\ell)) \in (c_k x_1 + \alpha'(x_2, \dots, x_\ell)).$$

Then since $\delta x_1 = 0$, $\delta\alpha'(x_2, \dots, x_\ell)$ is divisible by $c_k x_1 + \alpha'(x_2, \dots, x_\ell)$ for all $k = 1, \dots, m$. Hence is divisible by

$$\prod_{k=1}^{m}(c_k x_1 + \alpha'(x_2, \dots, x_\ell)).$$

Now we restrict to $x_1 = 0$, then $\delta|_{x_1=0}\alpha'$ is divisible by $(\alpha')^m$. Thus (i) is proved.

(ii) Let $\delta_1 = \theta_E, \delta_2, \dots, \delta_\ell$ be basis of $D(\mathcal{A})$ such that $\delta_2, \dots, \delta_\ell \in D_1(\mathcal{A})$. Then $\delta_2|_{x_1=0}, \dots, \delta_\ell|_{x_1=0}$ are linearly independent over $S/x_1S = \mathbb{C}[x_2, \dots, x_\ell]$. Furthermore, we have

$$\sum_{i=2}^{\ell} \operatorname{pdeg} \delta_i|_{x_1=0} = |\mathcal{A}| - 1 = \sum_{X \in \mathcal{A}^{H_1}} \mathbf{m}^{H_1}(X).$$

By Saito's criterion, they form free basis of $D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$. □

It seems natural to pay attention to the exact sequence

$$0 \longrightarrow D_1(\mathcal{A}) \xrightarrow{x_1\cdot} D_1(\mathcal{A}) \xrightarrow{\rho} D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}). \tag{7}$$

From the above proof, we know that if $\mathcal{A}$ is free, the restriction map ρ is surjective.

Corollary 1.34. *If the restriction map ρ is surjective, and $D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$ is free with exponents $(d_2, \dots, d_\ell)$, then $\mathcal{A}$ is free with exponents $(1, d_2, \dots, d_\ell)$.*

Proof. By the assumption, there exists $\delta_2, \ldots, \delta_\ell \in D_1(\mathcal{A})$ such that $\rho(\delta_2) = \delta_2|_{x_1=0}, \ldots, \rho(\delta_\ell) = \delta_\ell|_{x_1=0}$ are basis of $D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$. As in the previous proof, $\delta_2, \ldots, \delta_\ell$ and θ_E are linearly independent and the sum of pdeg is $|\mathcal{A}|$. Hence by Saito's criterion, $(\theta_E, \delta_2, \ldots, \delta_\ell)$ is the basis of $D(\mathcal{A})$. □

Generally, ρ is not surjective. However, local freeness implies local surjectivity.

Definition 1.35. *Let $\mathcal{A}$ be an arrangement and $H_1 \in \mathcal{A}$. Then $\mathcal{A}$ is said to be locally free along H_1 if $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}$ is free for all $X \in L(\mathcal{A})$ with $X \subset H_1$ and $X \neq 0$.*

Local freeness along H_1 implies

$$0 \longrightarrow D_1(\mathcal{A}_X) \xrightarrow{x_1 \cdot} D_1(\mathcal{A}_X) \xrightarrow{\rho} D(\mathcal{A}_X^{H_1}, \mathbf{m}_X^{H_1}) \longrightarrow 0$$

for all $X \in L(\mathcal{A}), X \neq 0$ with $X \subset H_1$. Thus we have an exact sequence of sheaves over $\mathbb{P}^{\ell-1}$.

$$0 \longrightarrow \widetilde{D_1(\mathcal{A})}(-1) \xrightarrow{x_1 \cdot} \widetilde{D_1(\mathcal{A})} \xrightarrow{\rho} D(\widetilde{\mathcal{A}^{H_1}, \mathbf{m}^{H_1}}) \longrightarrow 0. \tag{8}$$

Thus we obtain a relation between Ziegler's multirestriction and restriction of the sheaf $D_1(\mathcal{A})$.

Proposition 1.36. *If $\mathcal{A}$ is locally free along H_1, then*

$$\widetilde{D_1(\mathcal{A})}|_{H_1} = D(\widetilde{\mathcal{A}^{H_1}, \mathbf{m}^{H_1}}).$$

The above proposition combined with Proposition 1.19 and Horrocks criterion (Theorem 1.5 (2), see also subsequent Remark 1.6), we have the following criterion for freeness.

Theorem 1.37. *([53]) Assume that $\ell \geq 4$. Then $\mathcal{A}$ is free with exponents $(1, d_2, \ldots, d_\ell)$ if and only if the following conditions are satisfied.*

- *$\mathcal{A}$ is locally free along H_1, and*
- *Ziegler's multirestriction $(\mathcal{A}^{H_1}, m^{H_1})$ is free with exponents $(d_2, \ldots, d_\ell)$.*

The above criterion is not valid for $\ell = 3$. Indeed for $\ell = 3$, the both conditions are automatically satisfied. There exist non free 3-arrangements. For characterizing freeness of 3-arrangements, we need characteristic polynomials.

Theorem 1.38. *([54]) Let $\mathcal{A}$ be a 3-arrangement. Put $\chi(\mathcal{A}, t) = (t - 1)(t^2 - b_1 t + b_2)$ and $\exp(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}) = (d_1, d_2)$. Then*

(i) *$b_2 \geq d_1 d_2$, furthermore $b_2 - d_1 d_2 = \dim \operatorname{Coker}(\rho : D_1(\mathcal{A}) \to D(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}))$.*

(ii) *If $b_2 = d_1 d_2$, then $\mathcal{A}$ is free with exponents $(1, d_1, d_2)$.*

The proof is based on an analysis of Solomon-Terao's formula. Theorem 1.38 is also a corollary to a result in the next section (Theorem 1.44), a vector bundle version.

By combining Theorem 1.37 and 1.38, we recently obtained the following criterion for $\ell \geq 4$.

Theorem 1.39. *(Abe-Yoshinaga [11]) Assume that $\ell \geq 4$ and the multirestriction is free with $\exp(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}) = (d_2, \dots, d_\ell)$. Put $\chi(\mathcal{A}, t) = (t-1)(t^{\ell-1} - b_1 t^{\ell-2} + b_2 t^{\ell-3} - \dots)$. Then*

$$b_2 \geq \sum_{2 \leq i < j \leq \ell} d_i d_j,$$

and $\mathcal{A}$ is free if and only if the equality holds.

Remark 1.40. *At a glance, this result is similar to that of Bertone-Roggero (Theorem 1.5 (3)). However at this moment, I can not find any (simple) logical implications.*

Example 1.41. *Let $\mathcal{A} = \{H_1, \dots, H_{19}\}$ be the cone of G_2-Catalan arrangement $G_2^{[-1,1]}$ (see the separate page for figure). Using Abe's inequality (Theorem 1.27 (ii)) and Theorem 1.38, we can prove the freeness of $\mathcal{A}$ as follows. First the characteristic polynomial is*

$$\chi(\mathcal{A}, t) = (t-1)(t-7)(t-11).$$

Let us consider the multirestriction $(\mathcal{A}^{H_1}, \mathbf{m}^{H_1})$. Put the exponents $\exp(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}) = (d_1, d_2)$. Then by Theorem 1.38,

$$d_1 d_2 \leq 77.$$

Since the multirestriction is balanced, by Abe's inequality, we have

$$|d_1 - d_2| \leq 6 - 2 = 4.$$

Combining these two inequalities, we have $d_1 d_2 = 77$ hence $\mathcal{A}$ is free with exponents $(1, 7, 11)$.

We emphasize that in the above example, only the computation of characteristic polynomial is enough to prove freeness.

1.6. **Characteristic polynomials and Chern polynomials.** Let $\mathcal{A}$ be an arrangement in V of $\dim V = \ell$. By Terao's factorization theorem, if $\mathcal{A}$ is free with exponents $(d_1, \dots, d_\ell)$, then

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - d_i).$$

On the other hand, the sheafification splits $\widetilde{D(\mathcal{A})} = \mathcal{O}_{\mathbb{P}^{\ell-1}}(-d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{\ell-1}}(-d_1)$. The Chern polynomial of this sheaf is

$$(9) \qquad \begin{aligned} c_t(\widetilde{D(\mathcal{A})}) &= \sum_{i=1}^{\ell-1} c_i(\widetilde{D(\mathcal{A})}) t^i \\ &\equiv \prod_{i=1}^{\ell} (1 - d_i t) \mod t^\ell, \end{aligned}$$

where $c_i(-)$ is i-th Chern number. It is easily seen these two polynomials are related by the following formula:

$$(10) \qquad t^\ell \cdot \chi(\mathcal{A}, \frac{1}{t}) = c_t(\widetilde{D(\mathcal{A})}) \mod t^\ell.$$

Note that the left hand side of (10) is computed by Solomon-Terao's formula (Theorem 1.15). Mustaţă and Schenck proved that similar formula compute the Chern polynomial for arbitrary vector bundle on the projective space.

Theorem 1.42. *(Mustaţă and Schenck* [25]*) Let $\mathcal{E}$ be a vector bundle over $\mathbb{P}^n$ of rank r. Then*

$$c_t(\mathcal{E}) = \lim_{x \to 1} (-t)^r (1-x)^{n+1-r} \sum_{i=0}^{r} H(\Gamma_*(\bigwedge^i \mathcal{E}), x) \left(\frac{x-1}{t} - 1 \right)^i.$$

As a corollary, we have:

Corollary 1.43. *Let $\mathcal{A}$ be a locally free arrangement. Then the formula (10) holds.*

Using Mustaţă-Schenck, we can prove the following.

Theorem 1.44. *Let $\mathcal{E}$ be a rank two vector bundle on $\mathbb{P}^2$. Let $L \subset \mathbb{P}^2$ be a line. Put $\mathcal{E}|_L = \mathcal{O}_L(d_1) \oplus \mathcal{O}_L(d_2)$. Then $c_2(\mathcal{E}) \geq d_1 d_2$, furthermore*

$$c_2(\mathcal{E}) - d_1 d_2 = \dim \operatorname{Coker}(\Gamma_*(\mathcal{E}) \longrightarrow \Gamma_*(\mathcal{E}|_L)).$$

$\mathcal{E}$ is splitting if and only if $c_2(\mathcal{E}) = d_1 d_2$.

1.7. **Around Terao Conjecture.** In [46], Terao posed the following problem.

Problem 1.45. *Let $\mathcal{A}_1, \mathcal{A}_2$ be arrangements in V s. t. $L(\mathcal{A}_1) \simeq L(\mathcal{A}_2)$. Assume $\mathcal{A}_1$ is free. Then is $\mathcal{A}_2$ also free?*

It is obviously true in dimension 2. However the cases $\ell \geq 3$ are still open. In view of Theorem 1.38, it is true when the exponents of multirestriction is combinatorially determined for $\ell = 3$.

Proposition 1.46. *Let $\mathcal{A}_1, \mathcal{A}_2$ be in V of $\dim V = \ell = 3$ such that $L(\mathcal{A}_1) \simeq L(\mathcal{A}_2)$. Assume that $\mathcal{A}_1$ is free. If there exists a hyperplane $H \in \mathcal{A}$ such that the multirestriction $(\mathcal{A}^H, \mathbf{m}^H)$ satisfies one of conditions in Proposition 1.22, then $\mathcal{A}_2$ is also free.*

Thus the difficulty of Terao's conjecture for $\ell = 3$ is equivalent to the difficulty of determining exponents of 2-multiarrangements.

A possible approach to Terao's conjecture is to look at the set of arrangements which have prescribed intersection lattice, and then analyze the freeness on the set. We first introduce such set, the parameter space of arrangements having the fixed lattice. Let $\ell \geq 3$, $n \geq 1$. Fix a poset L. Then define the set $\mathcal{M}_{\ell,n}(L)$ of arrangements with lattice L by

$$\mathcal{M}_{\ell,n}(L) = \{\mathcal{A} = (H_1, \dots, H_n) \in (\mathbb{P}^{\ell-1*})^n \mid H_i \neq H_j, L(\mathcal{A}) \simeq L\}.$$

Terao's conjecture is equivalent to that the freeness/nonfreeness is preserved on $\mathcal{M}_{\ell,n}(L)$. Yuzvinsky proved that free arrangements form a Zarisky open subset in $\mathcal{M}_{\ell,n}(L)$.

Theorem 1.47. *(Yuzvinsky* [56, 57, 58]*)*

$$\{\mathcal{A} \in \mathcal{M}_{\ell,n}(L) \mid \mathcal{A} \textit{ is free } \}$$

is a Zarisky open subset of $\mathcal{M}_{\ell,n}(L)$.

In his proof, Yuzvinsky defined lattice cohomology using the structure of $L(\mathcal{A})$ and $D(\mathcal{A})$ (probably can be also formulated in terms of Grothendieck site). And he characterize the freeness of $\mathcal{A}$ via vanishings of these cohomology groups. The statement looks very similar to that of Horrocks (Theorem 1.5 (1)).

Problem 1.48. *Establish the relation between Yuzvinsky's and Horrocks' criteria for freeness.*

Here we recover (slite modified version of) Yuzvinsky's openness result for $\ell = 3$ by using upper semicontinuity of exponents of 2-multiarrangements. Similar to $\mathcal{M}_{\ell,n}(L)$, we introduce the following set of arrangements which have prescribed characteristic polynomial. Let $f(t) \in \mathbb{Z}[t]$.

$$\mathcal{C}_{\ell,n}(f) = \{\mathcal{A} = (H_1, \dots, H_n) \in (\mathbb{P}^{\ell-1*})^n \mid H_i \neq H_j, \chi(\mathcal{A}, t) = f(t)\}.$$

Theorem 1.49. *Let $Z \subset \mathcal{C}_{\ell,n}(f)$ be any algebraic subset. Then*

$$\{\mathcal{A} \in \mathcal{C}_{\ell,n}(f) \mid \mathcal{A} \textit{ is free}\}$$

is an Zarisky open subset of $\mathcal{C}_{\ell,n}(f)$.

Proof. By Terao's factorization theorem, if $f(t)$ is not split, then any $\mathcal{A} \in \mathcal{C}_{\ell,n}(f)$ is not free. We may assume that $f(t) = (t-1)(t-d_1)(t-d_2)$. Fix $H_1 \in \mathcal{A}$ and set $\exp(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}) = (d_1^{H_1}, d_2^{H_2})$. Then by Theorem 1.38, $|d_1^{H_1} - d_2^{H_2}| \geq |d_1 - d_2|$ and $\mathcal{A}$ is free if and only if the equality holds. By the upper semicontinuity (Proposition 1.26) of the difference $\Delta(\mathcal{A}^{H_1}, \mathbf{m}^{H_1}) = |d_1^{H_1} - d_2^{H_2}|$, the free locus $\{\mathcal{A} \mid \Delta < |d_1 - d_2| + \frac{1}{2}\}$ is a Zarisky open subset of $\mathcal{C}_{\ell,n}(f)$. □

Let L be a poset, and $f(t)$ be the corresponding characteristic polynomial. Then $\mathcal{M}_{\ell,n}(L) \subset \mathcal{C}_{\ell,n}(f)$. Applying Theorem 1.49 to $Z = \mathcal{M}_{\ell,n}(L)$, we obtain Yuzvinsky's openness for $\ell = 3$.

2. Weyl, Catalan and Shi arrangements

2.1. **Combinatorics of root systems.**

2.2. **Freeness of Catalan and Shi arrangements.**

3. K. Saito's theory of primitive derivation

3.1. **Summary of invariant theory of finite reflection groups.**

3.2. **Logarithmic vector fields.**

3.3. **The Primitive derivation.**

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In[42]:=

```
a = √3/2;
b = 1/2;
ContourPlot[{
  x == 0, x == -1, x == 1,
  (b + 1) x + a y == 0, (b + 1) x + a y == -1, (b + 1) x + a y == 1,
  b x + a y == 0, b x + a y == -1, b x + a y == 1,
  2 a y == 0, 2 a y == -1, 2 a y == 1,
  -b x + a y == 0, -b x + a y == -1, -b x + a y == 1,
  (-b - 1) x + a y == 0, (-b - 1) x + a y == -1, (-b - 1) x + a y == 1
 }, {x, -3.5, 3.5}, {y, -3.5, 3.5}]
```

Out[44]=