Artin groups and hyperplane arrangements

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Abstract

These are the notes of a mini-course given at the conference "Arrangements in Pyrénées", held in Pau (France) from 11th to 15th of June, 2012.

Definition. Let S be a finite set. A *Coxeter matrix* on S is a square matrix $M = (m_{s,t})_{s,t \in S}$ indexed by the elements of S and satisfying:

(a) $m_{s,s} = 1$ for all $s \in S$;

(b) $m_{s,t} = m_{t,s} \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for all $s, t \in S$, $s \neq t$.

A Coxeter matrix is usually represented by its *Coxeter graph*, $\Gamma = \Gamma(M)$. This is a labeled graph defined as follows.

(a) S is the set of vertices of Γ.

(b) Two vertices $s, t \in S$ are connected by an edge if $m_{s,t} \geq 3$, and this edge is labeled by $m_{s,t}$ if $m_{s,t} \geq 4$.

Example. The following matrix is a Coxeter matrix.

$$M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix} .$$

It is represented by the Coxeter graph drawn in Figure 1.

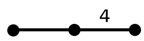

Figure 1. A Coxeter graph.

Definition. Let Γ be a Coxeter graph. The *Coxeter system* of Γ is defined to be the pair $(W, S) = (W_\Gamma, S)$, where S is the set of vertices of Γ and W is the group presented as follows.

$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{c} s^2 = 1 \text{ for all } s \in S \\ (st)^{m_{s,t}} = 1 \text{ for all } s, t \in S,\ s \neq t,\ m_{s,t} \neq \infty \end{array} \right\rangle .$$

The group W_Γ is called *Coxeter group* of Γ.

Example. Consider the Coxeter graph $\mathbb{A}_n$ drawn in Figure 2. The Coxeter group of $\mathbb{A}_n$ has the presentation

$$\left\langle s_1, \dots, s_n \;\middle|\; \begin{array}{c} s_i^2 = 1 \text{ for } 1 \le i \le n \\ (s_i s_{i+1})^3 = 1 \text{ for } 1 \le i \le n-1 \\ (s_i s_j)^2 = 1 \text{ for } |i-j| \ge 2 \end{array} \right\rangle .$$

This is the group $\mathfrak{S}_{n+1}$ of permutations of $\{1, \dots, n+1\}$ (symmetric group), where s_i is the transposition $(i, i+1)$ for all $i \in \{1, \dots, n\}$.

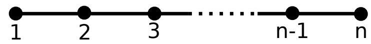

Figure 2. Coxeter graph $\mathbb{A}_n$.

Definition. If a, b are two letters and m is an integer ≥ 2, we set

$$\Pi(a, b, m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{if } m \text{ is even} \\ (ab)^{\frac{m-1}{2}} a & \text{if } m \text{ is odd} \end{cases}$$

Lemma 1. *Let Γ be a Coxeter graph. Then W_Γ has the following presentation.*

$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{c} s^2 = 1 \textit{ for all } s \in S \\ \Pi(s, t, m_{s,t}) = \Pi(t, s, m_{s,t}) \textit{ for all } s, t \in S,\ s \ne t,\ m_{s,t} \ne \infty \end{array} \right\rangle .$$

Proof. It suffices to prove that, for $m \ge 2$, the relation $(st)^m = 1$ is equivalent to the relation $\Pi(s, t, m) = \Pi(t, s, m)$ modulo the relations $s^2 = 1$ for all $s \in S$. We prove that for $m = 2$ and for $m = 3$. Assume $m = 2$.

$$(st)^2 = stst = 1 \;\Leftrightarrow\; st = t^{-1}s^{-1} = ts \;\Leftrightarrow\; \Pi(s, t, 2) = \Pi(t, s, 2) .$$

Assume $m = 3$.

$$(st)^3 = ststst = 1 \;\Leftrightarrow\; sts = t^{-1}s^{-1}t^{-1} = tst \;\Leftrightarrow\; \Pi(s, t, 3) = \Pi(t, s, 3) .$$

$\square$

Definition. Let Γ be a Coxeter graph, and let S be its set of vertices. Let $\Sigma = \{\sigma_s \mid s \in S\}$ be an abstract set in one-to-one correspondence with S. The *Artin system* of Γ is defined to be the pair $(A, \Sigma) = (A_\Gamma, \Sigma)$, where A is the group presented as follows.

$$A_\Gamma = \langle \Sigma \mid \Pi(\sigma_s, \sigma_t, m_{s,t}) = \Pi(\sigma_t, \sigma_s, m_{s,t}) \text{ for all } s, t \in S,\ s \ne t,\ m_{s,t} \ne \infty \rangle .$$

The group A_Γ is called *Artin group* of Γ. Thanks to Lemma 1, the map $\Sigma \to S$, $\sigma_s \mapsto s$, induces an epimorphism $\theta : A_\Gamma \to W_\Gamma$. The kernel of $\theta : A_\Gamma \to W_\Gamma$ is called *colored Artin group* of Γ and is denoted by CA_Γ.

Example. Consider the Coxeter graph $\mathbb{A}_n$ drawn in Figure 2. The Artin group of $\mathbb{A}_n$ has the presentation

$$\left\langle \sigma_1, \dots, \sigma_n \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \le i \le n-1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \ge 2 \end{array} \right\rangle .$$

This is the braid group $\mathcal{B}_{n+1}$ on $n+1$ strands. The colored Artin group of $\mathbb{A}_n$ is the pure braid group $\mathcal{PB}_{n+1}$.

The following is needed to understand the main results of the present mini-course.

Theorem 2 (Bourbaki [1]). *A Coxeter system (W_Γ, S) (resp. an Artin system (A_Γ, Σ)) entirely determines the Coxeter graph that defines it. In other words, given two Coxeter graphs Γ and Γ' with vertex sets S and S', respectively, the pairs (W_Γ, S) and $(W_{\Gamma'}, S')$ are isomorphic if and only if Γ and Γ' are isomorphic.*

Definition. Consider an nonempty open convex cone I in a finite dimensional real vector space V. A *hyperplane arrangement* in I is a (possibly infinite) family $\mathcal{H}$ of hyperplanes of V satisfying:

(a) $H \cap I \neq \emptyset$ for all $H \in \mathcal{H}$;

(b) $\mathcal{H}$ is *locally finite* in I, namely, for all $x \in I$, there is an open neighborhood U_x of x in I such that the set $\{H \in \mathcal{H} \mid H \cap U_x \neq \emptyset\}$ is finite.

Remark. In the "classical" definition of hyperplane arrangement I is the whole space V and $\mathcal{H}$ is finite.

Example. Set $V = \mathbb{R}^3$ and $I = \{(x, y, z) \in V \mid z > 0\}$. For $k \in \mathbb{Z}$, we denote by H_k the plane of V defined by the equation $x = kz$, and we denote by H'_k the plane defined by the equation $y = kz$. We set $\mathcal{H} = \{H_k,\ H'_k \mid k \in \mathbb{Z}\}$. This is a hyperplane arrangement in I. The trace of $\mathcal{H}$ on the affine plane of equation $z = 1$ is shown in Figure 3.

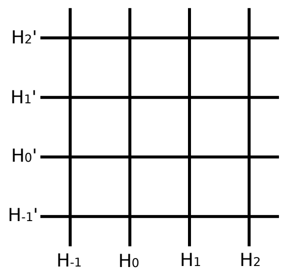

Figure 3. Hyperplane arrangement.

Definition. Let $V = \mathbb{R}^n$. A *reflection* on V is a linear transformation on V of order 2 which fixes a hyperplane. Note that there is no hypothesis on the orthogonality of the reflection, hence the fixed hyperplane does not necessarily determine the reflection. Let

$\overline{C}$ be a (closed) convex polyhedral cone with nonempty interior in V, and let C be the interior of $\overline{C}$. A *wall* of $\overline{C}$ is the support of a face (codimensional 1 facet) of $\overline{C}$, namely, the hyperplane of V spanned by the face. Denote by $H_1, \dots, H_\ell$ the walls of $\overline{C}$. For each $i \in \{1, \dots, \ell\}$ we take a linear reflection s_i which fixes H_i, and we denote by W the subgroup of $\mathrm{GL}(V)$ generated by $\{s_1, \dots, s_\ell\}$. Then W is called *linear reflection group* if $wC \cap C = \emptyset$ for all $w \in W \setminus \{1\}$.

Theorem 3 (Vinberg [9]). *Let W be a linear reflection group, and let*

$$\overline{I} = \bigcup_{w \in W} w\overline{C}\,.$$

Then the following statements hold.

(1) (W, S) is a Coxeter system, where $S = \{s_1, \dots, s_\ell\}$.

(2) $\overline{I}$ is a convex cone with nonempty interior.

(3) The interior I of $\overline{I}$ is stable under the action of W, and W acts properly discontinuously on I.

(4) Let $x \in I$ such that $W_x = \{w \in W \mid w(x) = x\}$ is not trivial. Then there is a linear reflection r in W such that $r(x) = x$.

Definition. The open convex cone I in Theorem 3 is called *Tits cone* of W.

Definition. Let W be a linear reflection group. Denote by $\mathcal{R}$ the set of reflections that belong to W. For $r \in \mathcal{R}$, we denote by H_r the hyperplane fixed by r. We set $\mathcal{H} = \{H_r \mid r \in \mathcal{R}\}$. By Theorem 3, $\mathcal{H}$ is a hyperplane arrangement in the Tits cone I. It is called *Coxeter arrangement* (associated with the linear reflection group W).

Definition. If I is an open convex cone in $V = \mathbb{R}^n$ and $\mathcal{H}$ is a hyperplane arrangement in I, we set

$$M(\mathcal{H}) = (I \times I) \setminus \left(\bigcup_{H \in \mathcal{H}} H \times H \right)\,.$$

This is a connected manifold of dimension $2n$. Note that, if $I = V = \mathbb{R}^n$, then ($\mathcal{H}$ is finite and)

$$M(\mathcal{H}) = \mathbb{C}^n \setminus \left(\bigcup_{H \in \mathcal{H}} H_\mathbb{C} \right)\,,$$

where, for $H \in \mathcal{H}$, $H_\mathbb{C} = \mathbb{C} \otimes H$ denotes the complexification of H. if W is a linear reflection group and $\mathcal{H}$ is its associated Coxeter arrangement, then we set $M(W) = M(\mathcal{H})$. Observe that W acts freely and properly discontinuously on $M(W)$. We set

$$N(W) = M(W)/W\,.$$

A detailed and complete proof of the following theorem will appear in the proceedings of the conference.

Theorem 4. *Let W be a linear reflection group. According to the above notations, we denote by S the set of reflections with respect to the walls of the cone $\overline{C}$. By Theorem 3, (W, S) is a Coxeter system. We denote by Γ its defining graph (which is unique by Theorem 2).*

(1) (Charney, Davis [2]) *The homotopy type of $N(W)$ depends only on Γ.*

(2) (Van der Lek [5]) *The fundamental group of $N(W)$ is the Artin group A_Γ of Γ.*

This theorem is related to one of the central questions in the theory of Artin groups, the $K(\pi, 1)$ *conjecture*, that we turn now to state.

Definition. Let G be a (discrete) group and let X be a connected CW-complex. We say that X is a $K(G, 1)$*-space* if $\pi_1(X) = G$ and the universal cover of X is contractible. In that case, we also say that X is *aspherical*. Recall that the cohomology of a group G coincides with that of any $K(G, 1)$-space.

Conjecture 5. ($K(\pi, 1)$ conjecture). *Consider a linear reflection group W, and keep the above used notations. Then $N(W)$ is a $K(A_\Gamma, 1)$-space.*

Example 1. Consider the symmetric group $\mathfrak{S}_{n+1}$ acting on the vector space $V = \mathbb{R}^{n+1}$ by permutation of the coordinates. If $w \in \mathfrak{S}_{n+1}$ and $x = (x_1, \dots, x_{n+1})$, then

$$w \cdot x = (x_{w^{-1}(1)}, \dots, x_{w^{-1}(n+1)})\,.$$

Let

$$\overline{C} = \{x \in V \mid x_1 \le x_2 \le \dots \le x_{n+1}\}\,.$$

For $i, j \in \{1, \dots, n+1\}$, $i \ne j$, we denote by $H_{i,j}$ the hyperplane defined by the equation $x_i = x_j$. Then $\overline{C}$ is a convex polyhedral cone with nonempty interior, and the walls of $\overline{C}$ are $H_{1,2}, H_{2,3}, \dots, H_{n,n+1}$. For $i \in \{1, \dots, n\}$, $s_i = (i, i+1)$ is a reflection with respect to $H_{i,i+1}$. The pair $(\mathfrak{S}_{n+1}, S)$ is the Coxeter system of $\mathbb{A}_n$, where $\mathbb{A}_n$ is the Coxeter graph drawn in Figure 2. In this case we have

$$\overline{I} = \bigcup_{w \in \mathfrak{S}_{n+1}} w\overline{C} = V\,.$$

So, $I = V$, too. The set $\mathcal{R}$ of reflections coincides with the set of transpositions, thus $\mathcal{H} = \{H_{i,j} \mid 1 \le i < j \le n+1\}$. This hyperplane arrangement is called *braid arrangement*. We identify $V \times V$ with $\mathbb{C}^{n+1} = \mathbb{C} \otimes V$. Then

$$M(\mathfrak{S}_{n+1}) = \mathbb{C}^{n+1} \setminus \left(\bigcup_{i<j} \mathbb{C} \otimes H_{i,j} \right)$$

is the *space of ordered configurations* of $n+1$ points in $\mathbb{C}$, and

$$N(\mathfrak{S}_{n+1}) = M(\mathfrak{S}_{n+1})/\mathfrak{S}_{n+1}$$

is the *space of (non-ordered) configurations* of $n+1$ points in $\mathbb{C}$. The fundamental group of $N(\mathfrak{S}_{n+1})$ is the braid group $\mathcal{B}_{n+1}$, and the fundamental group of $M(\mathfrak{S}_{n+1})$ is the pure braid group $\mathcal{PB}_{n+1}$. Moreover, we have the following.

Theorem 6 (Fadell, Neuwirth [4]). *The space $N(\mathfrak{S}_{n+1})$ is a $K(\mathcal{B}_{n+1}, 1)$-space.*

Proof. In order to prove Theorem 6, we need the following three results. These are classical and well-known in homotopy theory.

(1) Let $X \to Y$ be a covering map. Then X is aspherical if and only if Y is aspherical.

(2) Let $X \to B$ be a locally trivial fibration map with connected fiber F. If B and F are both aspherical, then X is aspherical as well.

(3) Any graph is aspherical.

By (1), in order to prove that $N(\mathfrak{S}_{n+1})$ is aspherical, it suffices to prove that $M(\mathfrak{S}_{n+1})$ is aspherical. We show that $M(\mathfrak{S}_{n+1})$ is aspherical by induction on n. Suppose $n = 1$. Then

$$M(\mathfrak{S}_2) = \mathbb{C}^2 \setminus \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = z_2\} \simeq \mathbb{C} \times \mathbb{C}^* \,.$$

The space $\mathbb{C}$ is aspherical because it is contractible. The circle is a deformation retract of $\mathbb{C}^*$ (see Figure 4), thus $\mathbb{C}^*$ has the same homotopy type as the circle, therefore $\mathbb{C}^*$ is aspherical by (3). By (2) we conclude that $M(\mathfrak{S}_2)$ is aspherical.

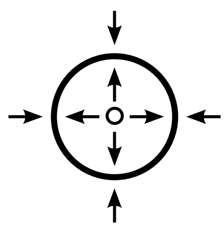

Figure 4. Retraction on a circle.

Suppose now that $M(\mathfrak{S}_n)$ is aspherical. It is shown (with some effort) that the map

$$\begin{array}{ccc} M(\mathfrak{S}_{n+1}) & \to & M(\mathfrak{S}_n) \\ (z_1, \dots, z_n, z_{n+1}) & \mapsto & (z_1, \dots, z_n) \end{array}$$

is a locally trivial fibration. The fiber above $(1, \dots, n)$ is

$$\{(1, \dots, n, z_{n+1}) \mid z_{n+1} \not\in \{1, \dots, n\}\} \simeq \mathbb{C} \setminus \{1, \dots, n\} \,.$$

Observe there is a graph which is a deformation retract of $\mathbb{C}\setminus\{1,\dots,n\}$ (see Figure 5), thus $\mathbb{C}\setminus\{1,\dots,n\}$ is aspherical by (3). We conclude by (2) that $M(\mathfrak{S}_{n+1})$ is aspherical. $\square$

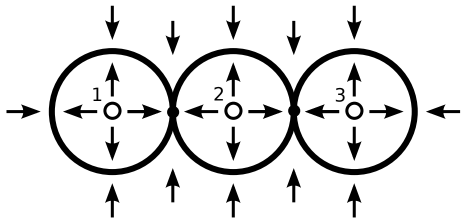

Figure 5. Retraction on a graph.

More generally, we have the following.

Theorem 7 (Deligne [3]). *If W is a finite linear reflection group, then $N(W)$ is aspherical.*

Example 2. Consider the Euclidean affine plane $\mathbb{E}^2$. For $k \in \mathbb{Z}$, we denote by D_k the affine line defined by the equation $x = k$, and by D'_k the line defined by the equation $y = k$ (See Figure 6). We denote by s_k the orthogonal reflection with respect to D_k, and we denote by s'_k the orthogonal reflection with respect to D'_k. We denote by W the affine orthogonal group generated by the s_k's and the s'_k's. The determination of the elements of W is left to the reader. However, we specify that, apart from reflections, W contains half-turns, translations, and slide-symmetries. It is easily proved that W is generated by s_0, s_1, s'_0, s'_1, and that W admits the following presentation.

$$W = \langle s_0, s_1, s'_0, s'_1 \mid s_0^2 = s_1^2 = {s'_0}^2 = {s'_1}^2 = 1\,,\ (s_0 s'_0)^2 = (s_0 s'_1)^2 = (s_1 s'_0)^2 = (s_1 s'_1)^2 = 1 \rangle\,.$$

This is the Coxeter group of the Coxeter graph drawn in Figure 7.

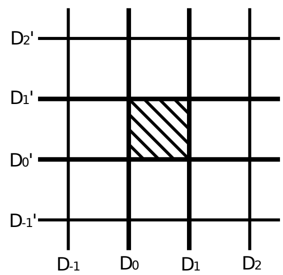

Figure 6. Grid lines in the affine plane.

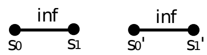

Figure 7. Coxeter graph.

We embed $\mathbb{E}^2$ into $\mathbb{R}^3$ via the map

$$(x, y) \mapsto (x, y, 1)$$

(see Figure 8). Let $\mathrm{Aff}(\mathbb{E}^2)$ denote the affine group of $\mathbb{E}^2$. Recall that, for every $f \in \mathrm{Aff}(\mathbb{E}^2)$, there are a unique linear transformation $f_0 \in \mathrm{GL}(\mathbb{R}^2)$ and a unique vector $u = u(f) \in \mathbb{R}^2$ such that $f = T_u \circ f_0$, where T_u denotes the translation with respect to the vector u. There is an embedding $\mathrm{Aff}(\mathbb{E}^2) \hookrightarrow \mathrm{GL}(\mathbb{R}^3)$ defined by

$$f \mapsto \begin{pmatrix} f_0 & u \\ 0 & 1 \end{pmatrix} .$$

Note that this embedding leaves invariant the space $\mathbb{E}^2$ embedded in $\mathbb{R}^3$ as above. Then we can consider W as a subgroup of $\mathrm{GL}(\mathbb{R}^3)$ via this embedding. For $k \in \mathbb{Z}$, we denote by H_k the linear plane spanned by D_k, and we denote by H_k' the linear plane spanned by D_k'. Then s_k is a linear reflection with fixed hyperplane H_k, and s_k' is a linear reflection with fixed hyperplane H_k'.

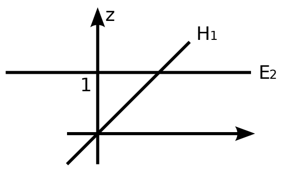

Figure 8. Embedding of $\mathbb{E}^2$ into $\mathbb{R}^3$.

Consider the square

$$\overline{C_0} = \{(x, y) \in \mathbb{E}^2 \mid 0 \le x, y \le 1\}$$

in $\mathbb{E}^2$. Denote by $\overline{C}$ the cone above $\overline{C_0}$. This is a convex polyhedral cone whose walls are H_0, H_1, H_0', H_1'. Observe that $wC \cap C = \emptyset$ for all $w \in W \setminus \{1\}$, thus W is a linear reflection group. It is easily checked that

$$\overline{I} = \bigcup_{w \in W} w\overline{C} = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\} \cup \{(0, 0, 0)\} ,$$

thus

$$I = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\} .$$

On the other hand,

$$\mathcal{H} = \{H_k, H_k' \mid k \in \mathbb{Z}\} .$$

Example 3. Let Γ be a Coxeter graph, and let (W, S) be its Coxeter system. We consider an abstract set $\{e_s \mid s \in S\}$ in one-to-one correspondence with S, and we denote by V the vector space with basis $\{e_s \mid s \in S\}$. Define the symmetric bilinear form $B : V \times V \to \mathbb{R}$ by

$$B(e_s, e_t) = \begin{cases} -\cos(\frac{\pi}{m_{s,t}}) & \text{if } m_{s,t} \neq \infty \\ -1 & \text{if } m_{s,t} = \infty \end{cases}$$

For all $s \in S$ we define $\rho_s \in \mathrm{GL}(V)$ by

$$\rho_s(x) = x - 2\, B(x, e_s) e_s , \quad x \in V .$$

It is easily checked that ρ_s is a reflection for all $s \in S$, and that the map $S \to \mathrm{GL}(V)$, $s \mapsto \rho_s$, induces a linear representation $\rho : W \to \mathrm{GL}(V)$. This representation is called *canonical representation* of (W, S).

We denote by V^* the dual space of V. Recall that every linear map $f \in \mathrm{GL}(V)$ determines a linear map $f^t \in \mathrm{GL}(V^*)$ defined by

$$\langle f^t(\alpha), x \rangle = \langle \alpha, f(x) \rangle \,,$$

for all $\alpha \in V^*$ and all $x \in V$. The *dual representation* $\rho^* : W \to \mathrm{GL}(V^*)$ of ρ is defined by

$$\rho^*(w) = (\rho(w)^t)^{-1} \,,$$

for all $w \in W$. For all $s \in S$ we set $H_s = \{\alpha \in V^* \mid \langle \alpha, e_s \rangle = 0\}$. On the other hand, we set

$$\overline{C} = \{\alpha \in V^* \mid \langle \alpha, e_s \rangle \geq 0 \text{ for all } s \in S\} \,.$$

Theorem 8 (Tits [8], Bourbaki [1]). *Let Γ be a Coxeter graph, and let (W, S) be its Coxeter system.*

(1) The canonical representation $\rho : W \to \mathrm{GL}(V)$ and its dual $\rho^* : W \to \mathrm{GL}(V^*)$ are faithful.

(2) The set $\overline{C}$ is a convex simplicial cone whose walls are H_s, $s \in S$. The transformation $\rho^*(s)$ is a linear reflection with fixed hyperplane H_s, for all $s \in S$. We have $\rho^*(w)C \cap C = \emptyset$ for all $w \in W \setminus \{1\}$.

Corollary 9. *$\rho^*(W)$ is a linear reflection group whose associated Coxeter system is $(\rho^*(W), \rho^*(S)) \simeq (W, S)$.*

From now on, we fix a Coxeter graph Γ, we denote by $(W, S) = (W_\Gamma, S)$ its Coxeter system, and we denote by $(A, \Sigma) = (A_\Gamma, \Sigma)$ its Artin system.

For $X \subset S$ we introduce the following notations.

- Γ_X is the full Coxeter subgraph of Γ generated by X.
- W_X is the subgroup of W generated by X.
- $\Sigma_X = \{\sigma_s \mid s \in X\}$, and A_X is the subgroup of A generated by Σ_X.

The subgroup W_X (resp. A_X) of W (resp. A) is called *standard parabolic subgroup*.

Theorem 10. *Let X be a subset of S.*

(1) (Bourbaki [1]) *(W_X, X) is the Coxeter system of Γ_X.*

(2) (Van der Lek [5]) (A_X, Σ_X) *is the Artin system of* Γ_X.

Example. Consider the Coxeter graph $\Gamma = \mathbb{B}_4$ drawn in the left-hand side of Figure 9. We denote by $S = \{s_1, s_2, s_3, s_4\}$ the set of vertices of Γ numbered according to the figure. Set $X = \{s_1, s_2, s_4\}$. Then Γ_X is the Coxeter graph $\mathbb{B}_2 \sqcup \mathbb{A}_1$ drawn in the right-hand side of Figure 9,

$$W_X = \langle s_1, s_2, s_4 \mid s_1^2 = s_2^2 = s_4^4 = 1\,,\ (s_1 s_2)^4 = (s_1 s_4)^2 = (s_2 s_4)^2 = 1 \rangle\,, \text{ and}$$
$$A_X = \langle \sigma_1, \sigma_2, \sigma_4 \mid \sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_2\sigma_1\sigma_2\sigma_1\,,\ \sigma_1\sigma_4 = \sigma_4\sigma_1\,, \sigma_2\sigma_4 = \sigma_4\sigma_2 \rangle\,.$$

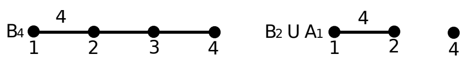

Figure 9. The Coxeter graphs $\mathbb{B}_4$ and $\mathbb{B}_2 \sqcup \mathbb{A}_1$.

Definition. For $w \in W$, we define the *length* of w, denoted by $\lg(w)$, to be the shortest length of an expression of w on the elements of S. Let X be a subset of S. We say that an element w in W is *X-minimal* if its length is minimal among the elements of the coset wW_X. On the other hand, we denote by $\mathcal{S}^f$ the set of subsets X of S such that W_X is finite. Note that we assume $\emptyset \in \mathcal{S}^f$ and $W_\emptyset = \{1\}$.

Lemma 11. *Let $\preceq$ be the relation on $W \times \mathcal{S}^f$ defined by*

$$(u, X) \preceq (v, Y)$$

if

$$X \subset Y\,,\ v^{-1}u \in W_Y\,,\ \text{and } v^{-1}u \text{ is } X\text{-reduced.}$$

Then $\preceq$ is a (partial) ordering relation.

Definition. A *simplicial complex* Υ is a pair $(V, E) = (V(\Upsilon), E(\Upsilon))$, where $V = V(\Upsilon)$ is a set, called *set of vertices*, and $E = E(\Upsilon)$ is a set of finite nonempty subsets of V, called *set of simplices*, satisfying:

(a) all the singletons are simplices;

(b) the intersection of any two simplices is a simplex.

From a simplicial complex Υ one can naturally construct a topological space gluing topological simplices. This space is called the *geometric realization* of Υ, and it is denoted by $|\Upsilon|$.

Definition. With an ordered set $(V, \preceq)$ one can associate a simplicial complex, denoted by V' and called *derived complex*, whose set of vertices is V itself, and whose simplices are the finite chains of V. The *Salvetti complex* of Γ, denoted by $\Omega(\Gamma)$, is defined to be the geometric realization of the derived complex of $(W \times \mathcal{S}^f, \preceq)$. Note that the action of W on $W \times \mathcal{S}^f$ defined by

$$w \cdot (u, X) = (wu, X)$$

preserves the order. Hence, it induces a action of W on $\Omega(\Gamma)$. It is easily seen that this action is free and properly discontinuous.

Theorem 12. (Salvetti [6, 7], Charney, Davis [2]). *Let W be a linear reflection group, and let Γ be the Coxeter graph determined by W as shown before.*

(1) There exists a homotopy equivalence $\Omega(\Gamma) \to M(W)$ equivariant under the actions of W.

(2) The fundamental group of $\Omega(\Gamma)/W$ is A_Γ.

This proves Theorem 5, that is:

Corollary 13. *Let W be a linear reflection group, and let Γ be the Coxeter graph associated to W as shown before.*

(1) (Charney, Davis [2]) *The homotopy type of $N(W)$ depends only on Γ.*

(2) (Van der Lek [5]) *The fundamental group of $N(W)$ is the Artin group A_Γ of Γ.*

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