Legendre functions and the theory of characteristics

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Abstract

We devise a framework encompassing the classical theory of characteristics and the theory valid in the convex case recently obtained by R.T. Rockafellar and P. Wolenski. It relies on a notion of transform introduced by I. Ekeland. It involves a class of functions called Ekeland functions which is large enough to encompass convex functions, concave functions and linear-quadratic functions, as well as the class of classical Legendre functions. We also introduce a class of functions called (generalized) Legendre functions which is not as large but which is stable under an extension of the Ekeland transform which is akin to the Fenchel-Legendre transform.

Dedicated to Ivar Ekeland on his sixtieth birthday

1 Introduction

The objective of the present paper consists in an attempt to encompass in a single framework two theories of characteristics for first order Hamilton-Jacobi equations: the classical one which assumes differentiability properties and is local and the convex theory of [27], [28]. For that purpose, we introduce a class of functions which are nonsmooth and nonconvex but retain the main duality property of the Legendre-Fenchel transform. This class, which contains the class of closed proper convex functions, classical Legendre functions and quadratic functions defined by a nondegenerated bilinear form appears in a natural way when using the Legendre duality introduced by Ekeland [8], [9]. However, in order to encompass the full generality of the conjugacy theory of convex analysis, we have to extend the process of [8], [9] and use closure processes. We perform such an extension by using a result in [22] which relies on the Ekeland variational principle and enables one to approach any element of the graph of a closed proper convex function by elements corresponding to points at which the function is subdifferentiable. Among the reasons justifying such an extension is the well known fact that the domain of the subdifferential of a closed proper convex function is non convex but its closure (and interior) is convex. Thus, this new class of functions we call generalized Legendre functions, instead of being more special than the class of convex functions as in [23, Chapter 26], [6], [4] is a
larger class. Using the methods of nonsmooth analysis, we also introduce Ekeland subsets and Legendre subsets of normed vector spaces (n.v.s.). On the other hand, we avoid the geometrical framework of Lagrangian submanifolds and contact manifolds present in [1], [8], [11], [16], [33]. We do not consider either questions of multi-valuedness of the transform which are dealt with in [8]. In fact, the notions we adopt impose single-valuedness of the transform.

Our approach can be seen as an instance of the methods of nonsmooth analysis which strive to encompass in a general framework both differential calculus and convex analysis. In our views, the most important feature of these developments is the “Copernican revolution” consisting in considering in a unified framework functions, sets and multimappings (correspondences or relations):

passage from sets to functions: to a subset $S$ of a n.v.s. one associates the distance function $d_S(\cdot) = \inf_{s \in S} d(\cdot, s)$ or the indicator function $\iota_S$ given by $\iota_S(x) = 0$ if $x \in S$, $+\infty$ else.

passage from functions to sets: to a function $f$ one associates its epigraph $E_f := \text{epi} f$.

passage from multifunctions to sets: take the graph $G(F) \subset X \times Y$ of the relation $F$.

The main tool is the notion of subdifferential which generalizes in a one-sided way the notion of derivative. It is not our purpose to present here a general definition of subdifferential; we refer to [13], [14] and their references for axiomatic studies of subdifferentials (which may vary with the purpose). Here we just assume that $\partial$ is a map which assigns to $f \in F(X)$ and $x \in \text{dom } f$ a subset $\partial f(x)$ of the topological dual $Y := X^*$ of $X$. We will mainly use the firm (or Fréchet) subdifferential $\partial^F$ given by

$$x^* \in \partial^F f(x) \iff \text{any } \varepsilon > 0, \ 0 \text{ is a local minimizer of } w \mapsto f(x + w) - \langle x^*, w \rangle + \varepsilon \|w\|$$

and the directional (or Dini-Hadamard, Hadamard or contingent) subdifferential $\partial^D$ given by

$$x^* \in \partial^D f(x) \iff \text{any } \varepsilon > 0, \ u \in X \setminus \{0\}, \ (0, u) \text{ is a local minimizer on } \mathbb{R}_+ \times X \text{ of the function } (t, v) \mapsto f(x + tv) - \langle x^*, tv \rangle + \varepsilon \|tv\|.$$ 

We also mention the limiting subdifferential $\partial^L$ defined through a limiting procedure from $\partial^F$. In the sequel, we also use the normal cone $N(E, x)$ to a subset $E$ of $X$ at $x \in E$ defined as $\partial E(x)$, where $\iota_E$ is the indicator function of $E$. It is easy to check that when $\partial$ is either $\partial^D$ or $\partial^F$, this definition coincides with the usual geometric notions. In particular, for $\partial = \partial^D$, the normal cone $N(E, x)$ is the polar of the tangent cone $T(E, x)$ to a subset $E$ of a n.v.s. $X$ at $x \in E$ is the set of $v \in X$ such that there is a sequence $((t_n, v_n)) \to (0_+, v)$ in $\mathbb{R} \times X$ satisfying $x + t_nv_n \in E$ for each $n \in \mathbb{N}$.

After some preliminaries recalling the classical theory of characteristics ([10], [15]) and the Rockafellar-Wolenski approach in the convex case, we deal with the Legendre transform as devised by Ekeland and we specify the classes of functions we consider. We mention some examples and properties and we relate these classes to classes of sets. We display an explicit formula for the solution of a first order Hamilton-Jacobi equation with data satisfying assumptions related to these classes of functions and we end our study with a connection with the theory of characteristics.
Other approaches to the theory of characteristics are given in the recent monographs [17], [19], [29], [32].

2 Preliminaries: characteristics

Let us consider the first order partial differential equation

\[ F(x, Du(x), u(x)) = 0 \quad x \in \Omega \]
\[ u(x) = g(x) \quad x \in \partial \Omega \]

where \( \Omega \) is a smooth open subset of some Banach space \( X \), \( \partial \Omega \) is its boundary and \( F : \Omega \times X^* \times \mathbb{R} \to \mathbb{R}, g : \partial \Omega \to \mathbb{R} \) are given \( C^2 \) functions. In order that such an equation be solvable, some compatibility condition has to be satisfied. The classical theory of characteristics is a means to find a local solution around some point \( (x_0, y_0, z_0) \in \partial \Omega \times X^* \times \mathbb{R} \) satisfying \( F(x_0, y_0, z_0) = 0 \), \( g(x_0) = z_0 \), \( y_0 | T(x_0, \partial \Omega) = dg(x_0) \) by using the solution to the system of ordinary differential equations

\[ \hat{x}'(s) = D_y F(\hat{x}(s), \hat{y}(s), \hat{z}(s)) \]
\[ \hat{y}'(s) = -D_x F(\hat{x}(s), \hat{y}(s), \hat{z}(s)) - D_z F(\hat{x}(s), \hat{y}(s), \hat{z}(s)) \hat{y}(s) \]
\[ \hat{z}'(s) = D_y F(\hat{x}(s), \hat{y}(s), \hat{z}(s)) \hat{y}(s) \]

and the initial conditions

\[ \hat{x}(0) = x_0, \quad \hat{y}(0) = Dg(x_0), \quad \hat{z}(0) = g(x_0). \]

Suppose \( D_y F(x_0, y_0, z_0).v \neq 0 \) for some normal vector \( v \) to \( \partial \Omega \). Then performing a change of variables \( (x, t) \) instead of \( x \) one can assume that locally \( \Omega = X_0 \times (0, +\infty) \) where \( X_0 \) is an open subset of an hyperplane of \( X \) and that the equation is of the form

\[ D_t u(x, t) + H(x, t, D_x u(x, t), u(x, t)) = 0, \quad (x, t) \in X_0 \times (0, +\infty) \]
\[ u(x, 0) = g(x) \quad x \in X_0. \]

Then the characteristic system is transformed into the following one

\[ \hat{x}'(s) = D_y H(\hat{x}(s), \hat{y}(s), \hat{z}(s)) \]
\[ \hat{y}'(s) = -D_x H(\hat{x}(s), \hat{y}(s), \hat{z}(s)) - D_z H(\hat{x}(s), \hat{y}(s), \hat{z}(s)) \hat{y}(s) \]
\[ \hat{z}'(s) = D_y H(\hat{x}(s), \hat{y}(s), \hat{z}(s)) \hat{y}(s) - H(\hat{x}(s), \hat{y}(s), \hat{z}(s)). \]

**Theorem 2.1** Suppose that for some \( x_0 \in X \) and some \( t_0 > 0 \) the solution \( \hat{x}(t_0, \cdot) \) with initial data \( x_0, y_0 := Dg(x_0), z_0 = g(x_0) \) realizes a diffeomorphism of a neighborhood \( U \) of \( x_0 \) onto a neighborhood \( V \) of \( \hat{x}(t_0, x_0) \). Then the function \( (x, t) \mapsto \hat{z}(t, \hat{x}(t, \cdot)^{-1}(x)) \) is a solution of class \( C^2 \) to the Hamilton-Jacobi equation.

This result is particularly effective when dealing with quasi-linear equations. In the last section of the present paper we will deal with the case when \( H \) does not depend on \( x, t, z \); then the characteristic system takes a particularly simple form.
3 Characteristics in convex analysis

In two remarkable papers, [27], [28], Rockafellar and Wolenski have studied the Hamilton-Jacobi equation

\[ D_t u(x,t) + H(x, D_x u(x,t)) = 0, \quad (x,t) \in X \times (0, +\infty) \]
\[ u(x,0) = g(x) \quad x \in X \]

where \( g \) is a given closed proper convex function and \( H \) is a finite concave-convex function on \( X \times X^* \) (with \( X \) finite dimensional) satisfying the following growth conditions for some real numbers \( \alpha, \beta, \gamma, \delta \) and some finite convex functions \( \varphi, \psi \)

\[ H(x,y) \leq \varphi(y) + (\alpha \| x \| + \beta) \| x \| \quad \forall (x,y) \in X \times X^*, \]
\[ H(x,y) \geq \psi(x) - (\gamma \| x \| + \delta) \| y \| \quad \forall (x,y) \in X \times X^*. \]

They associate to it the Bolza problem

\[ \text{minimize } g(w(0)) + \int_0^t L(w(s), w'(s)) ds : w(\cdot) \in W^{1,1}([0,t], X), \; w(t) = x \]

in which \( L(x, \cdot) = H(x, \cdot)^* \) and prove that its value \( v(x,t) \) is a solution to the above Hamilton-Jacobi equation in the sense that

\[ (p,q) \in \partial^L v(x,t) \iff p \in \partial v(t, \cdot)(x), \; q = -H(x,p) \]

where \( \partial^L \) is the limiting subdifferential and \( \partial \) is either \( \partial^L \) or the contingent or Hadamard subdifferentials or the subdifferential of convex analysis. In these papers culminates the duality theory for convex Bolza problems performed in the seventies ([24]-[26]). Moreover, they give a global version of the theory of characteristics. They observe that for a given convex function \( f : X \to \mathbb{R} := \mathbb{R} \cup \{\infty\} \) its subjet

\[ J^- f := \{(x,y,z) \in X \times X^* \times \mathbb{R} : y \in \partial f(x), \; z = f(x)\} \]

is a Lipschitzian submanifold of \( X \times X^* \times \mathbb{R} \) and that the flow associated with the characteristic equations

\[ \widehat{x}'(t) \in \partial H(\widehat{x}(t), \cdot)(\widehat{y}(t)) \]
\[ \widehat{y}'(t) \in \partial (-H)(\cdot, \widehat{y}(t))(\widehat{x}(t)) \]
\[ \widehat{z}'(t) = \langle \widehat{x}'(t), \widehat{y}(t) \rangle - H(\widehat{x}(t), \widehat{y}(t)) \]

carries the subjet \( J^- g \) of \( g \) onto the subjet \( J^- v(t, \cdot) \) of \( v(t, \cdot) \).

It is our purpose here to consider such questions under relaxed convexity assumptions.
4 The Ekeland and Legendre transforms

Let $X$ and $Y$ be two sets paired by a coupling function $c : X \times Y \to \mathbb{R}$. What we call here the Ekeland transform is a simple transform for multimappings $F : X \rightrightarrows Y \times \mathbb{R}$ (identified with their graphs $G(F)$ whenever there is no risk of confusion) associating to $F$ the multimapping $F^E : Y \rightrightarrows X \times \mathbb{R}$ given by

$$F^E(y) := \{(x,s) \in Y \times \mathbb{R} : (y, c(x,y) - s) \in F(x)\}.$$ 

In terms of graphs one has

$$G(F^E) = \{(y,x,s) \in Y \times X \times \mathbb{R} : (y, x, c(x,y) - s) \in G(F)\},$$

so that $G(F^E)$ is the image of $G(F)$ by the mapping $L : X \times Y \times \mathbb{R} \to Y \times X \times \mathbb{R}$ given by $L(x,y,z) := (y,x,c(x,y) - z)$. This transform is involutive:

$$(F^E)^E = F,$$

as easily checked. This transform has a special interest when $Y$ has a base point $0_Y$ and when $c(x,0_Y) = 0$ for each $x \in X$. Then, if $x$ is a critical point of $F$ in the sense that there exists some $r \in \mathbb{R}$ such that $(0_Y, r) \in F(x)$, one also has $(x, -r) \in F(0_Y)$.

When $X$ is a normed vector space, and when one disposes of a subdifferential $\partial$, this transform can be specialized to functions $f : X \to \mathbb{R}^* := \mathbb{R} \cup \{\infty\}$ in the class $\mathcal{F}(X)$ of functions on which $\partial$ is defined. For this purpose, one can associate to $f$ its hypergraph or (first order) subjet $J^0 f : X \rightrightarrows Y \times \mathbb{R}$ (or $\partial$-subjet, to be more precise) given by

$$J^0 f(x) := \{(y,r) : y \in \partial f(x), r = f(x)\}.$$ 

Note that when $f$ is of class $C^k$ ($k \geq 2$), $J^0 f = J^1 f$, the one-jet of $f$, is a Lagrangian submanifold of class $C^{k-1}$ of $X \times X^* \times \mathbb{R}$: the pull-back to $J^1 f$ of the differential form $\omega$ given by $\omega(x,y,z,u,v,w) = w - \langle y,u \rangle$ is null. For a nonsmooth function $f$ one may observe that the subjet $J^0 f$ is a super-Lagrangian subset of $X \times X^* \times \mathbb{R}$ in the sense that $\omega(x,y,z,u,v,w) \geq 0$ for any $(x,y,z) \in J^0 f$ and any $(u,v,w) \in T(J^0 f,(x,y,z))$. Note that the image $M' = L(M)$ of a super-Lagrangian subset $V$ of $X \times X^* \times \mathbb{R}$ is a sub-Lagrangian of $X^* \times X \times \mathbb{R}$ in the sense that $\omega'(y',x',z',v',u',w') \leq 0$ for any $(y',x',z') \in M'$ and any $(v',u',w') \in T(M',(y',x',z'))$. We shall not make use of these remarks in the sequel.

**Definition 4.1** A function $f : X \to \mathbb{R}$ is an Ekeland function (for $\partial$ and $c$) if for any $y$ in the image $\partial f(X)$ of $\partial f$ in $Y := X^*$ one has

$$x_1, x_2 \in (\partial f)^{-1}(y) \Rightarrow c(x_1,y) - f(x_1) = c(x_2,y) - f(x_2).$$ 

Then the Ekeland transform of $f$ is the function $f^E$ given by $f^E(y) := c(x,y) - f(x)$ for $y \in \partial f(X)$, $x \in (\partial f)^{-1}(y)$, $f^E(y) := +\infty$ for $y \in Y \setminus \partial f(X)$. 

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Condition (1) means that the restriction of the projection mapping \((y, s) \mapsto y\) to the projection of \((J^\partial f)^E\) on \(Y \times \mathbb{R}\) is injective. It can be interpreted as follows: it requires that \((J^\partial f)^E\) is an *hypergraph* in the sense that it is a subset of \(Y \times X \times \mathbb{R}\) of the form

\[
\bigcup_{y \in Y} \{y\} \times G(y) \times \{g(y)\}
\]

for some mappings \(G : Y \rightrightarrows X, g : Y \to \mathbb{R}\) such that \(\text{dom } G = \text{dom } g = \partial f(X)\). It is natural to consider the function \(y \mapsto f^E(y) := g(y)\) as a transform of \(f\); we call it the Ekeland transform of \(f\). Before going any further, let us give some examples.

**Example 1.** Let \(\mathcal{F}(X)\) be the set of convex functions on \(X\) and let \(\partial\) be the subdifferential of convex analysis or any other subdifferential coinciding with it on \(\mathcal{F}(X)\), \(c\) being the usual pairing of \(X\) with its dual space \(Y := X^*\). Then, for any \(f \in \mathcal{F}(X), y \in Y, x \in (\partial f)^{-1}(y)\) means that the function \(w \mapsto f(w) - \langle w, y \rangle\) attains its minimum at \(x\). Thus, condition (1) is fulfilled and \(f^E(y)\) coincides with \(f^*(y) := \sup_{x \in X} \left(\langle w, y \rangle - f(w)\right)\), the value of the conjugate function of \(f\).

**Example 2.** Let \(\mathcal{F}(X)\) be the set of *paraconvex* (or *semiconvex*) functions on \(X\) with respect to some differentiable function \(k : X \to \mathbb{R}\), i.e. the set of functions \(f\) such that \(f + k\) is convex. Let us take the pairing \(c_k : X \times Y \to \mathbb{R}\) given by \(c_k(x, y) := \langle x, y \rangle - k(x)\) and the subdifferential \(\partial_k\) given by

\[
y \in \partial_k f(x) \iff \forall w \in X \quad f(w) - c_k(w, y) \geq f(x) - c_k(x, y),
\]

or, equivalently, \(y \in \partial(f + k)(x), \partial\) being the subdifferential of convex analysis. Then, as easily checked, \(f\) is an Ekeland function for \(\partial_k\) and \(c_k\), and \(f^E(y) = \sup\{c_k(x, y) - f(x) : x \in X\} = (f + k)^*(y)\).

**Example 3.** Let \(\mathcal{F}(X)\) be the set of concave functions on \(X\) and let \(\partial\) be either the Fréchet or the Hadamard subdifferential, \(c\) being the usual pairing of \(X\) with its dual space \(Y := X^*\). Then, for any \(f \in \mathcal{F}(X), y \in Y, x \in (\partial f)^{-1}(y)\) means that \(f\) is Fréchet or Hadamard differentiable at \(x\); then the function \(w \mapsto f(w) - \langle w, y \rangle\) attains its maximum at \(x\). Thus, condition (1) is fulfilled and \(f^E(y)\) coincides with \(f_*(y) := \inf_{x \in X} \left(\langle w, y \rangle - f(w)\right)\), the value of the concave conjugate function of \(f\).

**Example 4.** Let \(\mathcal{F}(X)\) be the set of linear-quadratic functions on \(X\), i.e. the set of functions \(f\) given by \(f(x) := \frac{1}{2} \langle Ax, x \rangle - \langle b, x \rangle + c\) for some continuous symmetric linear map \(A : X \to Y := X^*, b \in Y, c \in \mathbb{R}\). Let \(\partial\) be either the Fréchet or the Hadamard subdifferential (or the corresponding differentiation operator). Then \(f\) is an Ekeland function. In fact, given \(y \in Y, x_1, x_2 \in X\) such that \(f'(x_i) = y\) one has

\[
\langle x_i, y \rangle - f(x_i) = \langle Ax_i - b, x_i \rangle - \frac{1}{2} \langle Ax_i, x_i \rangle + \langle b, x_i \rangle - c = \frac{1}{2} \langle Ax_i, x_i \rangle - c
\]

and

\[
\langle Ax_1, x_1 \rangle - \langle Ax_2, x_2 \rangle = \langle A(x_1 - x_2), x_1 \rangle + \langle Ax_2, x_1 - x_2 \rangle = 0
\]
since $A$ is symmetric and $Ax_1 = y + b = Ax_2$. Thus, we can write $f^E(y) = \frac{1}{2}(y+b, A^{-1}(y+b)) - c$, even if $A$ is non invertible.

**Example 5.** Let $X := U \times V$ be a product of two n.v.s. and let $Y := U^* \times V^*$. Let $f : X \to \mathbb{R}$ be such that, for each $v \in V$, $f(\cdot, v)$ is an Ekeland function on $U$ and for each $u \in U$, $f(u, \cdot)$ is an Ekeland function on $V$. Then setting $(u^*, v^*) \in \partial f(u, v)$ iff $u^* \in \partial f(\cdot, v)(u)$ and $v^* \in -\partial(-f)(u, \cdot)(v)$, we see that $f$ is an Ekeland function on $X$. Note that when $f$ is convex-concave, $(u^*, v^*) \in \partial f(u, v)$ iff $(u, v)$ is a saddle point of $f - u^* \circ p_U - v^* \circ p_V$, where $p_U$ and $p_V$ are the canonical projections of $X$ onto $U$ and $V$ respectively. When $X$ is finite dimensional, the decomposition of quadratic forms on $X$ shows that the preceding example is a special case of the present example.

**Example 6.** Let $W$ be a normed vector space (or one of its open subsets) and let $g : W \to \mathbb{R}$ be differentiable. Then, $f : W \times (0, +\infty) \to \mathbb{R}$ given by $f(w, t) = tg(t^{-1}w)$ is an Ekeland function since for any $(w, t) \in W \times (0, +\infty)$ one has

$$\langle w, D_w f(w, t) \rangle + tD_t f(w, t) - f(w, t) = \langle w, Dg(\frac{w}{t}) \rangle + tg(\frac{w}{t}) - t^2t^{-2}Dg(\frac{w}{t}).w - tg(\frac{w}{t}) = 0.$$  

This last example shows that the domain of the Ekeland transform may be very small and that the Ekeland transform of an Ekeland function is not necessarily an Ekeland function. We now turn to a remedy to these ailments.

The next example is a refinement of the classical notion of Legendre function of class $C^k$. Recall that that notion enables one to pass from the Euler-Lagrange equations of the calculus of variations to the Hamilton equations which are explicit (rather than implicit) differential equations of first order (instead of second order). Let us give a precise definition in which we say that a mapping $g : U \to V$ between two metric spaces is stable or is Stepanovian if for any $\overline{u} \in U$ there exist some $r > 0$, $k \in \mathbb{R}_+$ such that for every $u$ in the ball $B(\overline{u}, r)$ of center $\overline{u}$ and radius $r$ one has

$$|g(u) - g(\overline{u})| \leq kd(u, \overline{u}).$$

**Definition 4.2** A function $f : U \to \mathbb{R}$ on an open subset $U$ of a normed vector space $X$ is a classical Legendre function if it is differentiable, if its derivative $f' : U \to Y := X^*$ is a Stepanovian bijection onto an open subset $V$ of $Y$ whose inverse $h$ is also Stepanovian.

Then one defines the Legendre transform of $f$ as the function $f^L : V \to \mathbb{R}$ given by

$$f^L(y) := \langle h(x), y \rangle - f(h(y)) \quad y \in V.$$  

It coincides with the Ekeland transform of $f$. Since $h$ is just a Stepanov function, it is surprising that $f^L$ is in fact of class $C^1$ (and of class $C^k$ when $f$ is of class $C^k$).

**Lemma 4.3** If $f$ is a classical Legendre function on $U$, then it is an Ekeland function and its Legendre transform is of class $C^1$ on $V := f'(U)$. Moreover $f^L$ is a classical Legendre function, $(f^L)^L = f$ and

$$v = Df(u) \Leftrightarrow u = Df^L(v) \quad \forall (u, v) \in U \times V.$$  

Furthermore, when $f$ is of class $C^k$, $f^L$ is of class $C^k$.  

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Proof. Given \( v := Df(u) \in V \), let \( y \in V - v \), let \( x := h(v + y) - h(v) \in U - u \) and let \( r(x) = f(u + x) - f(u) - Df(u)x \). Then, since \( h(v) = u \), \( h(v + y) = u + x \), one has

\[
\begin{align*}
f^L(v + y) - f^L(v) - \langle u, y \rangle &= \langle u + x, v + y \rangle - f(u + x) - \langle u, v \rangle + f(u) - \langle u, y \rangle \\
&= \langle x, v + y \rangle - Df(u)x - r(x) \\
&= \langle x, y \rangle - r(x).
\end{align*}
\]

Since there exists \( k \in \mathbb{R}_+ \) such that \( \|x\| \leq k \|y\| \) for \( \|y\| \) small enough, the last right hand side is a remainder, i.e. a \( o(y) \). Thus \( f^L \) is differentiable at \( v \) and \( Df^L(v) = u = h(v) \).

When \( f \) is of class \( C^k \), \( Df^L = h \) is of class \( C^{k-1} \) as an induction shows, thanks to the Stepanov property of \( f' \) and \( h \). \( \square \)

Remark. If in the preceding result one replaces the assumption that \( h = f' \) is a Stepanov homeomorphism with a Stepanov inverse by the assumption that \( h \) is a Stepanov open map with Stepanov local inverses, and if \( f \) is an Ekeland function, one gets that \( f^E = f^L \) is of class \( C^1 \) (and of class \( C^k \) if \( f \) is of class \( C^k \)) and is in fact a classical Legendre function.

One may wonder to what extent the definition of Ekeland functions depends on the choice of the subdifferential. It would be interesting to find a satisfactory answer in the case of the limiting subdifferential associated with a given subdifferential. We just present an obvious observation.

**Proposition 4.4** If a function \( f : X \to \mathbb{R} \cdot \) is an Ekeland function for a subdifferential \( \partial \), then it is an Ekeland function for any smaller subdifferential.

Thus, the class of Ekeland functions for the limiting subdifferential or for the Clarke subdifferential is more reduced than the class of Ekeland functions for the Fréchet or the proximal subdifferentials. Taking for \( \partial \) the Fenchel-Moreau subdifferential of convex analysis yields the class of all functions.

### 4.1 Legendre functions and Legendre transform

The following definition stems from our wish to get a symmetric concept. It is also motivated by the convex case in which the domain of \( f^E \) is the image of \( \partial f \) which is not necessarily convex, while a natural extension of \( f^E \) is the Fenchel conjugate whose domain is convex and which enjoys nice properties (lower semicontinuity, local Lipschitz property on the interior of its domain...).

**Definition 4.5** Let \( X \) and \( Y \) be normed vector spaces paired by a coupling function \( c \). A l.s.c. function \( f : X \to \mathbb{R} \cdot \) is said to be a (generalized) Legendre function for a subdifferential \( \partial \) if there exists a l.s.c. function \( f^L : Y \to \mathbb{R} \) such that

(a) \( f \) and \( f^L \) are Ekeland functions and \( f^L | \partial f(X) = f^E | \partial f(X) \);  
(b) for any \( x \in \text{dom} f \) there is a sequence \((x_n, y_n, r_n)_n \) in \( J^0 f \) such that \((x_n, \langle x_n - x, y_n \rangle, r_n) \to (x, 0, f(x))\);
(b') for any \( y \in \text{dom} f^L \) there is a sequence \( (y_n, x_n, s_n) \) in \( f^0 f^L \) such that \( (y_n, (x_n, y_n - y), s_n) \to (y, 0, f^L(y)) \);
(c) the relations \( x \in X, y \in \partial f(x) \) are equivalent to \( y \in Y, x \in \partial f^L(y) \).

Condition (b) ensures that \( f \) is determined by its restriction to \( \text{dom} \partial f \). In fact, for any \( x \in \text{dom} f \) one has

\[
 f(x) = \lim_{x' \to x} \inf_{x' \in \text{dom} \partial f} f(x')
\]

since \( f(x) \leq \liminf_{x' \to x} f(x') \) and (b) implies \( f(x) = \lim_n f(x_n) \) for some sequence \( (x_n) \to x \) in \( \text{dom} \partial f \). Similarly, \( f^L \) is determined by its restriction to \( \text{dom} \partial f^L = \partial f(X) \). Conditions (b’) and (c) imply that \( f^L \) is determined by \( f \).

Condition (b) can be simplified when \( f \) is locally Lipschitzian on its domain, or, more generally, when \( f \) is locally tranquil on \( \text{dom} \partial f \) in the following sense: for any \( x_0 \in \text{dom} f \) there exist \( r > 0 \) and \( c \geq 0 \) such that

\[
 \liminf_{(t,u) \to (0+,u)} \frac{1}{t} (f(x + tv) - f(x)) \leq c \quad \forall x \in B(x_0, r) \cap \text{dom} \partial f, \forall u \in B(0,1).
\]

This condition is milder than local quietness of \( f \) (i.e. local calmness of \( -f \)). It clearly implies that for any \( x \in B(x_0, r) \cap \text{dom} f \) and any \( y \in \partial f(x) \) one has \( \|y\| \leq c \). Therefore in that case condition (b) is equivalent to the simpler condition

(b0) for any \( x \in \text{dom} f \) there exists a sequence \( (x_n)_n \) in \( \text{dom} \partial f \) such that \( (x_n, f(x_n)) \to (x, f(x)) \).

A similar observation holds for condition (b’). The interest of the stringent conditions (b) and (b’) is to make the extensions as close as possible to \( f \) and \( f^E \) respectively.

The preceding definition is symmetric. Indeed, setting \( (f^L)^L = f \), for any \( x \in \partial f^L(Y) = \text{dom} \partial f \) (by (c)), we see that \( (f^L)^L(x) = f(x) = \langle x, y \rangle - f^E(y) = (f^L)^E(x) \) for \( y \in \partial f(x) \).

Let us give some examples of Legendre functions.

**Proposition 4.6** (a) Any classical Legendre function is a (generalized) Legendre function.

(b) Any l.s.c. proper convex function is a (generalized) Legendre function.

**Proof.** The first assertion is obvious: in condition (b) one takes \( (x_n, y_n, r_n) = (x, f'(x), f(x)) \) and we make a similar choice in (b’). The second assertion is a consequence of [22], taking for \( f^L \) the Fenchel conjugate \( f^* \) of \( f \). \( \Box \)

**Proposition 4.7** Let \( X \) be an Asplund space whose dual space \( X^* \) is also an Asplund space. Let \( f \) be a continuous concave function on an open convex subset of \( X \) such that the domain of the concave conjugate \( f_* \) of \( f \) is open. Then \( f \) is a (generalized) Legendre function for the Fréchet and the Hadamard subdifferentials.
Proof. By definition of an Asplund space, $f$ is Fréchet (sub-) differentiable on a dense subset of $\text{dom } f$. Given $x \in \text{dom } f$, taking $(x_n)$ in $\text{dom } f'$ with limit $x$ and using the local Lipschitz property of $f$ we see that condition (b) is satisfied. Similarly, assumption (b') is fulfilled. □

The following result uses the key fact of nonsmooth analysis that condition (b) of Definition 4.5 is satisfied for the Fréchet subdifferential on spaces having smooth bump functions.

**Proposition 4.8** Let $X$ be a Banach space such that $X$ and $X^*$ have $C^4$ smooth bump functions. Let $f$ be a lower semicontinuous Ekeland function whose Ekeland transform is a lower semicontinuous Ekeland function. Then $f$ is a (generalized) Legendre function for the Fréchet subdifferential.

**Example 7.** Let $\mathcal{F}(X)$ be the set of partially quadratic functions on $X$, i.e. the set of functions $f$ given by $f(x) := \frac{1}{2}\langle A(x-a), x-a \rangle - \langle b, x-a \rangle + c$ when $x$ belongs to some closed affine subspace $W + a$ of $X$ (with $W$ a vector subspace of $X$, $a \in X$) and $f(x) = +\infty$ when $x \in X \setminus (W + a)$, where $A : W \to W^*$ is some continuous symmetric linear map, $b \in Y := X^*$, $c \in \mathbb{R}$. Let $\partial$ be either the Fréchet or the Hadamard subdifferential. Then $f$ is an Ekeland function. In fact, given $y \in Y, x_1, x_2 \in X$ such that $y \in \partial f(x_i)$ one has $y | W = A(x_i - a) - b | W$ and

$$
\langle y, x_i \rangle - f(x_i) = \langle y, a \rangle + \langle A(x_i - a) - b, x_i - a \rangle - \frac{1}{2} \langle A(x_i - a), x_i - a \rangle + \langle b, x_i - a \rangle - c
$$

and

$$
\langle A(x_1 - a), x_1 - a \rangle - \langle A(x_2 - a), x_2 - a \rangle = \langle A(x_1 - x_2), x_1 - a \rangle + \langle A(x_2 - a), x_1 - x_2 \rangle = 0
$$

since $A$ is symmetric and $A(x_1 - a) = (y + b) | W = A(x_2 - a)$. Moreover, when $A$ is invertible, or, more generally when $A(W)$ is closed, what occurs when $W$ is finite dimensional, $f^E$ is also a partially quadratic function. In that case, we see that $\text{dom } \partial f = \text{dom } f$ and that the domain of $f^E$ is the set of $y \in Y$ such that $(y - b) | W \in A(W)$, so that $f$ is a Legendre function with $f^L = f^E$.

### 4.2 Operations on Ekeland and Legendre functions

In the present subsection $X$ is a n.v.s. with dual $Y$ and we take the Hadamard or the Fréchet subdifferential. The compatibility of the usual operations with respect to the concepts we study is not as rich as what occurs for the Fenchel transform, as simple examples show for the sum. However, some simple properties can be devised, in particular for the infimal convolution $\Box$ which is defined by

$$(g \Box h)(x) := \inf \{g(u) + h(v) : u, v \in X, u + v = x\}$$

for two functions $f, g$ on $X$. The infimal convolution $g \Box h$ is said to be exact at $x \in X$ if there exists some $u, v \in X$ such that $(g \Box h)(x) = g(u) + h(v)$. 


Proposition 4.9  (a) If $f$ is an Ekeland function, then for any positive real number $\lambda$ the function $\lambda f$ is an Ekeland function and $(\lambda f)^E(y) = \lambda f^E(\lambda^{-1}y)$. Moreover, if $f$ is a Legendre function, then $\lambda f$ is a Legendre function.

(b) If $g$ and $h$ are Ekeland functions, and if the infimal convolution $f := \square g h$ is exact, then $f$ is an Ekeland function and $(\square g h)^E = g^E + h^E$.

(c) If $W_0$, $X_0$ are open subsets of Banach spaces $W, X$ respectively, if $h : X_0 \to W_0$ is a mapping of class $C^1$ whose derivative at every point of $X_0$ is surjective, if $h$ is the restriction of a positively homogeneous mapping between $X$ and $W$ and if $g : W_0 \to \mathbb{R}$ is an Ekeland function, then $f := g \circ h$ is an Ekeland function provided that for any $x_1, x_2 \in X_0$ and $z_1, z_2 W^*$ such that $z_i \in \partial g(h(x_i))$ for $i = 1, 2$, $z_1 \circ h'(x_1) = z_2 \circ h'(x_2)$ one has $z_1 = z_2$. In particular, if $g$ is a continuous linear form and $h$ is as above, or if $A : X \to W$ is a surjective linear continuous map and if $g : W \to \mathbb{R}$ is an Ekeland function, then $f := g \circ A$ is an Ekeland function; in such a case, for any $y \in X^*$, $z \in W^*$ with $y = A^*(z)$, one has $f^E(y) = g^E(z)$.

(d) Suppose $W_0$, $X_0$ are open subsets of Banach spaces $W, X$ respectively, $h : X_0 \to W_0$ is a mapping of class $C^1$ which is the restriction of a positively homogeneous mapping between $X$ and $W$ and $g : W \to \mathbb{R}$ is a closed proper convex function such that

$$h'(x)(X) - \mathbb{R}_+(\text{dom } g - h(x)) = W$$

for each $x \in X_0$. Then $f := g \circ h$ is an Ekeland function provided that for any $x_1, x_2 \in X_0$ and $z_1, z_2 \in W^*$ such that $z_i \in \partial g(h(x_i))$ for $i = 1, 2$, $z_1 \circ h'(x_1) = z_2 \circ h'(x_2)$, one has $z_1 = z_2$.

(e) If $A : X \to W$ is an isomorphism and if $g : W_0 \to \mathbb{R}$ is a classical Legendre function on an open subset $W_0$ of $W$, then $f := g \circ A$ is a classical Legendre function on $X_0 := A^{-1}(W_0)$.

(f) If $f_i : X_i \to \mathbb{R}$ is an Ekeland (resp. Legendre) function for $i = 1, \ldots, k$, then $f$ given by $f(x) := f_1(x_1) + \ldots + f_k(x_k)$ for $x := (x_1, \ldots, x_k)$ is an Ekeland (resp. Legendre) function.

Proof. (a) Let $x_i \in X$ be such that $y \in \partial (\lambda f)(x_i)$ for $i = 1, 2$. Then $\lambda^{-1}y \in \partial f(x_i)$ and

$$\langle x_i, y \rangle - \lambda f(x_i) = \lambda \left( \langle x_i, \lambda^{-1}y \rangle - f(x_i) \right) = \lambda f^E(\lambda^{-1}y), \quad i = 1, 2,$$

so that $\lambda f$ is an Ekeland function and $(\lambda f)^E(y) = \lambda f^E(\lambda^{-1}y)$. When $f$ is a Legendre function, setting $(\lambda f)^L(y) = \lambda f^L(\lambda^{-1}y)$, we easily get that $\lambda f$ is a Legendre function.

(b) Let $y \in \partial f(X)$. Pick $x \in (\partial f)^{-1}(y)$. Since the infimal convolution is exact at $x$, there exist $u, v \in X$ such that $u + v = x$, $g(u) + h(v) = f(x)$. Then, one easily checks that $y \in \partial g(u) \cap \partial h(v)$. It follows that $\langle y, x \rangle - f(x)$ does not depend on the choice of $x \in (\partial f)^{-1}(y) :$

$$\langle y, x \rangle - f(x) = \langle y, u \rangle - g(u) + \langle y, v \rangle - h(v) = g^E(y) + h^E(y).$$

(c) Using the fact that $h$ is open at a linear rate at each point of $X_0$, one can easily check that for any $x \in X_0$ one has $\partial f(x) = \partial g(h(x)) \circ h'(x)$. Moreover, by the Euler relation, one has $h'(x).x = h(x)$. Then if $y \in \partial f(x_i)$ for $i = 1, 2$, one can find $z_i \in \partial g(h(x_i))$ such that $y = z_i \circ h'(x_i)$. Our assumption ensures that $z_1 = z_2$, and the Euler relation ensures that $h'(x_i).x_i = h(x_i)$, so that

$$\langle y, x_i \rangle - f(x_i) = \langle z_i \circ h'(x_i), x_i \rangle - g(h(x_i)) = \langle z_i, h(x_i) \rangle - g(h(x_i)) = g^E(z_i)$$

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is independent of $i : f$ is an Ekeland function. When $g$ is a continuous linear form the relations $z_i \in \partial g(h(x_i))$ imply that $z_i = g$. When $h$ is a surjective continuous linear map, the relation $z_1 \circ h'(x_1) = z_2 \circ h'(x_2)$ amounts to $z_1 \circ h = z_2 \circ h$, hence $z_1 = z_2$ since $h$ is surjective.

(d) It is known that the qualification condition we assume implies that $\partial f(x) = \partial g(h(x)) \circ h'(x)$ both for the firm and the directional subdifferentials (see for instance [21], Prop. 4.1).

The rest of the proof is similar to the preceding case.

(e) If $A : X \rightarrow W$ is an isomorphism and if $g : W_0 \rightarrow \mathbb{R}$ is a classical Legendre function, then $f := g \circ A$ is such that $x \mapsto f'(x) = g'(A(x)) \circ A$ is a $C^1$ diffeomorphism from $X_0 := A^{-1}(W_0)$ onto $A^T(g'(W_0))$. Then, for $x^* \in A^T(g'(W_0))$, $x \in X$ with $x^* = A^T(g'(Ax))$, one has $f^L(x^*) = \langle x, x^* \rangle - f(x) = \langle Ax, g'(Ax) \rangle - g(Ax) = g^L((A^T)^{-1}x^*)$.

(f) Since $f$ is a separable function, we have $y \in \partial f(x)$ if, and only if $y_i \in \partial f(x_i)$ for $i = 1, \ldots, k$ when $x := (x_1, \ldots, x_k)$, $y := (y_1, \ldots, y_k)$. The fact that $f$ is an Ekeland function when $f_i$ are Ekeland functions ensues and $f^E(y) = f^E_1(x_1) + \ldots + f^E_k(x_k)$. From this formula one sees that $f$ is a Legendre function when $f_1, \ldots, f_k$ are Legendre functions as $f^L$ is also separable.

## 5 Ekeland sets and relations

**Definition 5.1** A subset $E$ of a Banach space will be called an **Ekeland set** if its indicator function $\iota_E$ is an Ekeland function, i.e. if for any $x_1, x_2 \in E$ and any $y \in N(E, x_1) \cap N(E, x_2)$ one has $\langle x_1, y \rangle = \langle x_2, y \rangle$.

A subset $E$ of a Banach space will be called an **Legendre set** if its indicator function $\iota_E$ is an Legendre function.

In the following statement, we use the Fréchet subdifferential.

**Proposition 5.2** If the distance function $d_E$ to a subset $E$ of a normed vector space is an Ekeland function then $E$ is an Ekeland set.

Conversely, if a subset $E$ of a Hilbert space (or a reflexive Banach space with the Kadec-Klee property and a Gâteaux smooth norm off 0) $X$ is an Ekeland set, then its associated distance function $d_E$ is an Ekeland function.

**Proof.** Suppose $d_E$ is an Ekeland function. Let $y \in X^*$ and $x_1, x_2 \in E$ be such that $y \in \partial^F \iota_E(x_i)$ for $i = 1, 2$. Let $u \in X^*$ with norm one such that $y = \|y\| \cdot u$. Then, since by a well known result one has

$$\partial^F d_E(x) = N^F(E, x) \cap B_{X^*},$$

for each $x \in E$, where $B_{X^*}$ is the unit ball of $X^*$, and since the Fréchet normal cone $N^F(E, x)$ is $\partial^F \iota_E(x)$, one has

$$u \in \partial^F d_E(x_i) \quad i = 1, 2.$$ 

Since $d_E$ is an Ekeland function, one has

$$\langle x_1, u \rangle = \langle x_1, u \rangle - d_E(x_1) = \langle x_2, u \rangle - d_E(x_2) = \langle x_2, u \rangle$$

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It follows that \( \langle x_1, y \rangle = \|y\| \langle x_1, u \rangle = \|y\| \langle x_2, u \rangle = \langle x_2, y \rangle \) and \( E \) is an Ekeland subset of \( X \).

Conversely, let \( E \) be an Ekeland subset of \( X \). Let \( y \in X^* \backslash \{0\} \) and let \( x \in X \) be such that \( y \in \partial d_E(x) \). Then, by [5, Lemma 6] there exists some \( u \in E \) such that \( d_E(x) = \|x - u\| \) and \( y \in S(x - u) \) where
\[
S(x - u) := \{ y \in S_{X^*} : \langle x - u, y \rangle = \|x - u\| \}.
\]

Moreover \( y \in N(E, u) \). Then,
\[
\langle x, y \rangle - d_E(x) = \langle x, y \rangle - \|x - u\| = \langle x, y \rangle - \langle x - u, y \rangle = \langle u, y \rangle = \iota_E^E(y)
\]
is independent of the choice of \( x \in (\partial d_E)^{-1}(y) \). Now if \( 0 \in \partial d_E(x) \), then \( x \in E \) and \( \langle x, y \rangle - d_E(x) = 0 \). Thus \( d_E \) is an Ekeland function. \( \square \)

**Example 8.** Let \( E \) be the epigraph of the function \( x \mapsto -x^2 \) from \( \mathbb{R} \) to \( \mathbb{R} \). Then Proposition 4.7 and the following proposition shows that \( E \) is an Ekeland set. It uses the next elementary lemma.

**Lemma 5.3** If \( E \) is the epigraph of a lower semicontinuous function \( f : X \to \mathbb{R} \) and if \((x^*, -t^*) \in N(E, (x, t)) \) with \( t^* > 0 \), then \( x^*/t^* \in \partial f(x) \) and \( t = f(x) \).

**Proof.** Assume \( t > f(x) \). Then \((0, -1) \in T(E, (x, t)) \), as easily seen. Then the relation
\[
t^* = \langle (x^*, -t^*), (0, -1) \rangle \leq 0
\]
leads to a contradiction. \( \square \)

**Proposition 5.4** If the epigraph \( E \) of a function \( f \) is an Ekeland set then \( f \) is an Ekeland function and \( \iota_E^E(y, -s) = sf^E(s^{-1}y) = (sf)^E(y) \) for each \((y, s) \in X^* \times (0, +\infty) \); in particular \( f^E(y) = \iota_E^E(y, -1) \) for each \( y \in X^* \). Conversely, if \( f \) is a Stepanovian Ekeland function, then its epigraph is an Ekeland set.

**Proof.** Suppose \( E \) is an Ekeland set. Let \( x_1, x_2 \in X \), \( y \in X^* \) be such that \( y \in \partial f(x_1) \cap \partial f(x_2) \). Then, for \( i = 1, 2 \) one has \((y, -1) \in N(E, (x_i, f(x_i))) \), hence
\[
\langle x_1, y \rangle - f(x_1) = \langle x_1, f(x_1) \rangle, (y, -1) = \langle x_2, f(x_2) \rangle, (y, -1) = \langle x_2, y \rangle - f(x_2),
\]
and \( f \) is an Ekeland function. Moreover, by the preceding relations, one has \( f^E(y) = \iota_E^E(y, -1) \) for each \( y \in X^* \).

Conversely, let \( f \) be a locally Lipschitzian Ekeland function and let \((x_1, r_1), (x_2, r_2) \in E, (y, s) \in X^* \times \mathbb{R} \) be such that \((y, -s) \in N(E, (x_i, r_i)) \) for \( i = 1, 2 \). When \((y, -s) = (0, 0) \), we have \( \langle (y, -s), (x_i, r_i) \rangle = 0 \) for \( i = 1, 2 \). Because \( f \) is continuous, that happens whenever \( r_i > f(x_i) \) for some \( i \in \{1, 2\} \). Thus, when \((y, -s) \neq (0, 0) \) we have \( r_i = f(x_i) \) for \( i \in \{1, 2\} \), and since \( f \) is locally Lipschitz we cannot have \( s = 0 \). Therefore \( s > 0 \) (since \( (0, 1) \) is tangent to \( E \) at \((x_i, f(x_i)) \), so that \( s \in \mathbb{R}_+ \) and \( s^{-1}y \in \partial f(x_i) \), so that
\[
\langle (x_i, r_i), (y, -s) \rangle = s \langle x_i, s^{-1}y \rangle = sf^E(s^{-1}y).
\]
Therefore $E$ is an Ekeland set. We note that the assumption that $f$ is a Stepanovian function can be replaced by the assumption that $f$ is quiet (i.e. $-f$ is calm) at each point $x$ such that $N(E, (x, f(x)))$ is not $\{(0, 0)\}$. □

A notion of Ekeland relation can easily be deduced from the preceding analysis.

**Definition 5.5** A multimapping $F : X \rightrightarrows Y$ between two normed vector spaces is said to be an Ekeland (resp. Legendre) relation if its graph is an Ekeland (resp. Legendre) subset of $X \times Y$.

Then, for any $(x^*, y^*) \in X^* \times Y^*$ and any $(x_i, y_i) \in G(F)$ ($i = 1, 2$) such that $x^* \in D^*F(x_i, y_i)(y^*)$ where $D^*F$ is the coderivative of $F$, one has
\[
\langle x_1, x^* \rangle - \langle y_1, y^* \rangle = \langle x_2, x^* \rangle - \langle y_2, y^* \rangle,
\]

since $x^* \in D^*F(x_i, y_i)(y^*)$ means that $(x^*, -y^*) \in N(G(F), (x_i, y_i))$ for $i = 1, 2$.

### 6 The Lax formula and characteristics

Let us return to the Hamilton-Jacobi equation
\[
D_t u(x, t) + H(D_x u(x, t)) = 0, \quad (x, t) \in X \times (0, +\infty)
\]
\[
u(x, 0) = g(x) \quad x \in X
\]

where again $g$ and $H$ are given functions on $X$ and $X^*$ respectively. Let us assume that the epigraph $E$ of $H$ is a Legendre set and let us introduce the functions $F, G$ given by
\[
F(p, r) := \iota_E(p, -r) \quad (p, r) \in Y \times \mathbb{R},
\]
\[
G(x, t) := g(x) + \iota_{\{0\}}(t) \quad (x, t) \in X \times \mathbb{R}.
\]

Then let us set, for $(x, t) \in X \times \mathbb{R}_+$,
\[
u(x, t) = (F^{L\square}G)(x, t).
\]

**Lemma 6.1** The function $u$ given by the preceding formula coincides with the function $(x, t) \mapsto (g \Box (tH)^L)(x)$ on $X \times (0, +\infty)$. Moreover, for $(x, t) \in X \times (0, +\infty)$ the infimal convolution $(F^{L\square}G)(x, t)$ is exact if, and only if, the infimal convolution $(g \Box (tH)^*)(x)$ is exact. In particular, when $H$ is a closed proper convex function the preceding formula coincides with the Lax-Oleinik-Hopf formula given by
\[
u(x, t) := \inf_{w \in X} \sup_{p \in Y} (g(w) + \langle p, x - w \rangle - tH(p)) = (g \Box (tH)^*)(x).
\]
The assertion about exactness (i.e. attainment) ensues.

Moreover

\[
(F^L \Box G)(x,t) = \inf \{(sH)^L(x - w) + g(w) + \iota_{\{0\}}(t - s) : (w,s) \in X \times \mathbb{R}\}
\]

\[
= \inf \{(tH)^L(x - w) + g(w) : w \in X\} = (g \Box (tH)^L)(x).
\]

The assertion about exactness (i.e. attainment) ensues.

Since \(F\) is closed convex when \(H\) is a closed convex function, \(F^L = F^*\). Hence, for \((x,t) \in X \times \mathbb{R}_+\), one has

\[
F^L(x,t) = F^*(x,t) = \sup \{\langle x,p \rangle - rt : (p,r) \in E\}
\]

\[
= \sup \{\langle x,p \rangle - rt : p \in \text{dom} \ H, \ r \geq H(p)\} = (tH)^*(x).
\]

**Proposition 6.2** Let \(H\) be a function whose epigraph is a Legendre set and let \(g\) be an arbitrary lower semicontinuous function such that for each \(t > 0\) the infimal convolution \(u := g \Box (tH)^L\) is exact. Then \(u\) is a unilateral solution of equation (H-J) on \(X \times \mathbb{R}_+\) in the sense that for any \((x,t) \in X \times (0, \infty)\) and any \((p,q) \in \partial u(x,t)\) one has

\[
q + H(p) = 0.
\]

**Proof.** Let \((x,t) \in X \times (0, \infty)\) and let \((p,q) \in \partial u(x,t)\). Then \(p \in \partial u(\cdot,t)(x)\). Since the infimal convolution \(u := g \Box (tH)^L\) is exact there exists \(w \in X\) such that \(u(x,t) = (tH)^L(x-w) + g(w)\).

Then we have \(p \in \partial (tH)^L(x-w)\). Since \(tH\) is a Legendre function by Proposition 4.9 (a), we have \((x-w,t) \in \partial F(p,q)\) or \((x-w,-t) \in \partial L_E(p,-q)\). As \(t\) is positive, this inclusion means that \(t^{-1}(x-w) \in \partial H(p)\) and the relation \(-q = H(p)\) holds in view of Lemma 5.3. \(\square\)

The condition that the infimal convolution is exact is satisfied under lower semicontinuity and coercivity assumptions in the finite dimensional case.

Now let us tackle the links with the method of characteristics. In the present case, because \(H\) does not depend on \(x,t,z\), the characteristic system takes a simple form:

\[
\begin{align*}
\tilde{x}'(s) &\in \partial H(\tilde{y}(s)) \\
\tilde{y}'(s) &= 0 \\
\tilde{z}'(s) &= \langle \tilde{x}'(t), \tilde{y}(s) \rangle - H(\tilde{y}(s)).
\end{align*}
\]

with the initial conditions \(\tilde{x}(0) = w, \tilde{y}(0) = w^* \in \partial g(w), \tilde{z}(0) = g(w)\). Given \((w,w^*) \in \partial g, \) and \(v \in \partial H(w^*)\), a solution of this system is given by \(\tilde{x}(s) = w + sv, \tilde{y}(s) = w^*, \tilde{z}(s) = g(w) + s(\langle v, w^* \rangle - H(w^*))\). The following result extends [20, Theorem 2.2] from the case of a strictly convex Hamiltonian to the case of a generalized Legendre Hamiltonian; see [31], [30] for related results dealing with regularity properties. It is a partial extension of [27, Thm 2.4] since in that paper \(H\) also depends on \(x\).
Proposition 6.3 Let $H$ be a Legendre function and let $g$ be an arbitrary l.s.c. function such that for each $t > 0$ the infimal convolution $u := g \square (tH)^L$ is exact. For each $(x,t) \in X \times (0, \infty)$ such that $\partial u(\cdot, t)(x)$ is nonempty, there exist $w \in X$ and $w^* \in \partial g(w)$ such that the characteristic curve emanating from $(w, w^*, g(w))$ satisfies $\hat{x}(t) = x$, $\hat{y}(t) = w^*$, $\hat{z}(t) = u(x,t)$. In fact, the subject of $u(\cdot, t)$ is contained in the image of the subject of $g$ by the flow defined by the characteristic system.

Proof. Given $(x,t) \in X \times (0, \infty)$ and $w \in W$ such that $u(x,t) = (tH)^L(x-w) + g(w)$, for each $p \in \partial u(\cdot, t)(x)$, one has $p \in \partial(tH)^L(x-w) \cap \partial g(w)$. Then, since $tH$ is a Legendre function by Proposition 4.9 (a), one gets $t^{-1}(x-w) \in \partial H(p)$. Setting $w^* := p$, $v := t^{-1}(x-w)$ we see that $x$ is the value at $s = t$ of the characteristic curve $\hat{x} : s \mapsto w + sv$. Correspondingly, since $v \in \partial H(w^*)$

\[
\hat{z}(t) = g(w) + t((v, w^*) - H(w^*)) = g(w) + tH^L(v) = g(w) + tH^L(t^{-1}(x-w)) = g(w) + (tH)^L(x-w) = u(x,t).
\]

Thus $(x, p, u(x,t))$ is the image of $(w, w^*, g(w))$ by the flow $(\hat{x}, \hat{y}, \hat{z})$ at time $t$. \hfill $\square$

Example 9. Let $X$ be a Hilbert space, let $A, B$ be symmetric linear continuous operators from $X$ to $X$, $A$ being invertible, and let $g$ and $H$ be given by $g(x) = \frac{1}{2}(Bx \mid x)$, $H(p) = \frac{1}{2}(Ap \mid p)$. Then $H$ is a locally Lipschitz Legendre function and $H^L(x) = \frac{1}{2}(A^{-1}x \mid x)$. Then the function $u$ given by

\[
u(x,t) := (g \square (tH)^L)(x) = \frac{1}{2}(B(I + tAB)^{-1}x \mid x)
\]

is a unilateral solution of equation (H-J) on $X \times (0, T)$ with $T := \|AB\|^{-1}$.

Example 10. Let $X = \mathbb{R}^2$ let $g$, $H$ be given by $g(x_1, x_2) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2$, $H(p_1, p_2) = \frac{1}{4}p_1^4 - \frac{1}{4}p_2^4$. Then $g$ and $H$ are classical Legendre functions and

\[
H^L(x_1, x_2) = \frac{3}{4}x_1^{4/3} - \frac{3}{4}x_2^{4/3}.
\]

Let $r(s, t)$ be the solution of the equation $r^3 + t^{-1/3}r - s = 0$. Then $u$ given by

\[
u(x_1, x_2, t) = \frac{1}{2}(x_1 - r(x_1, t)^{1/3})^2 + \frac{3}{4}t^{-1/3}(x_1, t)^{4/9} - \frac{1}{2}(x_2 - r(x_2, t)^{1/3})^2 - \frac{3}{4}t^{-1/3}(x_2, t)^{4/9}
\]

is a unilateral solution of equation (H-J) on $X \times (0, +\infty)$.

References


