Overdispersion and underdispersion characterization of weighted Poisson distributions

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Abstract

Consider a weighted Poisson distribution (WPD) having the same mean than the original Poisson distribution. This note proves that the mean value of the weight function with respect to the underlying Poisson distribution is a logconvex (logconcave) function if and only if the WPD is overdispersed (underdispersed) relative to the Poisson distribution. The logconvexity (logconcavity) of the weight function is only a sufficient condition, and some other properties are examined. For a comprehensive relation between over- and underdispersion, we then introduce a notion of duality between two WPD such that the product of their weight functions is one. Some illustrative examples are presented and discussed, in particular for practical weight functions.

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1 Introduction

The Poisson distribution has been considered as the “natural” distribution for count data. However it is equidispersed, that is the mean is equal to the variance. For many observed count data, the sample variance is greater or smaller than the sample mean. These phenomena are respectively called *overdispersion* and *underdispersion* relative to the Poisson distribution. They are well-known in the statistical literature and related domains. They may be due to one or more possible causes, such as heterogeneity and aggregation for overdispersion and repulsion for underdispersion although less frequent. Their studies are needed the use of differents modified Poisson distributions. Among recent papers, the reader is referred to Gelfand and Dalal (1990), Gupta *et al.* (2004), Hougaard *et al.* (1997), Kokonendji *et al.* (2004) and Whalin and Paris (2002) for only overdispersion. For both phenomena, we can consult Bosch and Ryan (1998), Cameron and Johansson (1997), Castillo and Pérez-Casany (1998, 2005), Dossou-Gbété *et al.* (2005), Luceño (2005), Puig and Valero (2005), Ridout and Besbeas (2004) and Shmueli *et al.* (2005).

The *weighted Poisson distributions* (WPD) are modified Poisson ones, which provide a unifying approach for both situations of over- and underdispersion. The concept of weighted distributions has been introduced by the method of ascertainment (Fisher, 1934; Rao, 1965); see Patil (2002) for more references. Let X be a nonnegative random variable with probability mass function $p(x;\theta)$, where $\theta \in \Theta$ is the natural parameter. Suppose that when the event $X = x$ occurs the probability of ascertaining it is $w(x)$. The recorded x is a realization of the random variable X^w, which is called the weighted (or ascertained) version of X. Its probability mass function is written as

$$p_w(x;\theta) = \frac{w(x)\; p(x;\theta)}{\mathbb{E}_\theta[w(X)]} = \Pr(X^w = x), \tag{1}$$

where $\mathbb{E}_\theta(\cdot)$ denotes the mean value with respect to the distribution of X depending of θ. The weight (or recording) function $w(x)$ is a nonnegative function and, from (1), we obviously have $0 < \mathbb{E}_\theta[w(X)] < \infty$. This function $w(x) = w(x;\phi)$ can depend on a parameter ϕ representing the recording mechanism. Note that w can be also connected to the natural parameter θ. Weighted distributions have been widely used as a tool in the selection of appropriate models for observed data drawn without a proper frame. The most usual weight function is $w(x) = x$, which provides the size-biased (or length-biased) version of the original random variable (e.g. Patil and Rao, 1978).

Concerning the particular case of the WPD and the closely problems of both over- and underdispersion relative to the Poisson distribution, the main goal of this paper is to provide a characterization of the weight function $w(x)$ following the over- and underdispersion. In the more recent paper of Castillo and Pérez-Casany (2005), they proposed a partial solution by considering the exponential weight function

$$w(x) = \exp\{r\ t(x)\}, \tag{2}$$

where $r \in \mathbb{R}$ and $t(\cdot)$ is a convex function. They showed that if $r > 0$ ($r = 0$ and $r < 0$) then the corresponding WPD is overdispersed (equidispersed and underdispersed) Poisson distribution. This result is already an extension of the previous one (Castillo and Pérez-Casany, 1998) with $t(x) = -\log(x+\alpha)$ in (2) for $\alpha \geq 0$. We are also motivated by the recent paper of Ridout and Besbeas (2004), which built an empirical model of WPD for underdispersion situation where $w(x) = \exp\{r|x-\mu|\}$ is centred on the mean μ of the underlying Poisson distribution. For econometric literature, we observe that Cameron and Johansson (1997) used the weight function of the polynomial form

$$w(x) = \left(1 + \sum_{k=1}^{p} \alpha_k\ x^k\right)^2, \quad \alpha_k \in \mathbb{R};$$

its corresponding WPD contains as special case the WPD of Castillo and Pérez-Casany (1998) with $r = 2p$ and $t(x) = \log(x+\alpha)$ in (2).

In section 2 we present the necessary and sufficient condition on a given weight function $w(x) = w(x;\phi)$ such that the corresponding WPD is overdispersed (underdispersed) relative to the Poisson distribution. The condition turns to be the logconvexity (logconcavity) of the function $\mu \mapsto \mathbb{E}_\mu[w(X)]$, where X is the underlying Poisson random variable with mean $\mu > 0$. Certain properties are investigated. In particular, we also show a sufficient condition from the logconvexity (logconcavity) of the weight function $x \mapsto w(x;\phi,\mu)$ which could be depended on the natural parameter μ. That leads to the notion of duality between two WPD, which can be used to construct another WPD with respect to one or both properties of over- and underdispersion. In section 3 we illustrate these results by some examples of the literature, in particular the COM-Poisson distribution of Shmueli *et al.* (2005) with $w(x) = (x!)^{1-\gamma}$. We also discuss the situation for which the weight function is connected to the parameter μ of the underlying Poisson distribution. The last section is devoted to the proofs.

2 Results

If it is necessary, we distinguish the cases $w(x) = w(x;\phi)$ to $w(x) = w(x;\phi,\mu)$ where ϕ is some free parameter and $\mu = \exp(\theta)$ is the (natural) parameter of the underlying Poisson distribution. We first state the main result.

Theorem 1 *Let X be the Poisson random variable with mean $\mu > 0$ and let*

$$w(x) = w(x;\phi) \tag{3}$$

a non null weight function. Consider X^w the corresponding weighted Poisson random variable such that

$$\mathbb{E}(X^w) = \mathbb{E}_\mu(X) = \mu. \tag{4}$$

Then the two following assertions are equivalent:
(i) the function $\mu \mapsto \mathbb{E}_\mu[w(X)]$ is logconvex (logconcave),
(ii) X^w is overdispersed (underdispersed) with respect to X.

Note that the equidispersion of X^w corresponds to $w(x) = 1, \forall x \in \mathbb{N}$, that is to the standard Poisson random variable X. For the proof of Theorem 1, we need the following lemma.

Lemma 2 *Under the assumption of Theorem 1, the characteristical variance of X^w is*

$$Var(X^w) = \mu\left(1 + \mu\,\frac{d^2}{d\mu^2}\log \mathbb{E}_\mu[w(X)]\right).$$

We then present a soft but more helpful version of Theorem 1. The next theorem is a kind of reformulation of the Castillo and Pérez-Casany (2005, Corollary 4) result, and its proof is really different and simple.

Theorem 3 *Under the assumption of Theorem 1, excepted that (3) is replaced by*

$$w(x) = w(x;\phi,\mu). \tag{5}$$

Then, X^w is overdispersed (underdispersed) relative to X if the weight function $x \mapsto w(x;\phi,\mu)$ is logconvex (logconcave).

Here is a joint consequence to the above Theorem 1 and Theorem 3. Its proof is omitted for brevity.

Corollary 4 *Under the assumption of Theorem 1, if the weight function $x \mapsto w(x;\phi) = w(x)$ is logconvex (logconcave) then $\mu \mapsto \mathbb{E}_\mu[w(X)]$ is also logconvex (logconcave).*

From the approach of Gupta *et al.* (1997), a clear property between the logconvexity or logconcavity of the weight function and the one of the associated WPD is given by the following proposition. Note that a discrete logconcave distribution is necessarily increasing failure rate (IFR) and strongly unimodal whereas a discrete logconvex distribution is necessarily decreasing failure rate (DFR) and infinitely divisible (e.g. Steutel, 1985). If a distribution is infinitely divisible it is overdispersed; if it is not infinitely divisible it can be either overor underdispersed. We recall that the Poisson distribution is logconcave (and hence IFR and strongly unimodal); it is infinitely divisible and equidispersed.

Proposition 5 *Let w be a positive weight function. If $x \mapsto w(x)$ is logconcave then its associated WPD is logconcave.*

We now introduce the notion of *(punctual) duality* for two WPD.

Definition 6 *Let w_1 and w_2 be two positive weight functions generating two WPD. These two WPD are said to be (punctually) dual if their weight functions satisfy*

$$w_1(x)w_2(x) = 1, \;\; \forall x \in \mathbb{N}. \tag{6}$$

As example of pair of dual WPD, we consider the WPD family of Castillo and Pérez-Casany (1998) such that $w_1(x) = (x+\alpha)^r$ and $w_2(x) = (x+\alpha)^{-r}$ with $\alpha, r \geq 0$. The standard Poisson distribution is *self-dual* because of its weight function $w(x) = 1, \forall x \in \mathbb{N}$.

Remark 7 *(a) The positivity of w_i in Definition 6 implies that the supports of the corresponding WPD are the whole nonnegative set $\mathbb{N}$.*
(b) Given a WPD generated by a positive weight function w. Then, its dual exists if and only if $\mathbb{E}_\mu[1/w(X)] < \infty$ where X is the underlying Poisson random variable.

The next proposition gives two useful properties of this duality.

Proposition 8 *(i) Let X^w be a weighted version of the Poisson random variable X with mean $\mu > 0$ and positive weight function w. Then, its dual exists if there exists $0 \leq c_0 < 1$ such that*

$$\lim_{x\to\infty} \frac{\mu \; w(x-1)}{x \; w(x)} = c_0.$$

(ii) If the pair of weighted Poisson random variables (X^{w_1}, X^{w_2}) is a pair of duals and such that one of w_i is logconvex (or logconcave). Then this pair is constituted of overdispersed and underdispersed Poisson random variables.

We finally conclude by observing that this notion of (punctual) duality can be extended to the *mean duality* as follows: replace in Definition 6 the positivity of w_i by the one of $\mathbb{E}_\mu[w_i(X)]$ and also the condition (6) by

$$\mathbb{E}_\mu[w_1(X)] \; \mathbb{E}_\mu[w_2(X)] = 1. \tag{7}$$

Many previous properties of punctual duality would be removed to the mean duality from w_i to $\mathbb{E}_\mu[w_i(X)]$. Thus, any WPD such that $\exists x \in \mathbb{N}, w(x) = 0$ can have its mean dual. For the moment, the mean duality is yet an uncommon notion in practice even for simple distributions such as the zero-truncated Poisson distribution with $w(0) = 0$, $w(x) = 1, x > 0$, and the Bernoulli distribution with $w(0) = w(1) = 1$, $w(x) = 0, x > 1$ (see also the size-biased Poisson distribution in the next section).

3 Some illustrations and discussions

We here point out some interesting situations with respect to the results of the previous section. This is done through the size-biased Poisson distribution, COM-Poisson distribution and some WPD generated by weight functions depending on natural parameter of the relevant Poisson distribution.

3.1 Size-biased Poisson distribution

The weight function used here is

$$w(x) = x.$$

Its corresponding WPD is known to be a 1-*translated Poisson distribution* (Patil and Rao, 1978), abbreviated to SBPD. The function $\mu \mapsto \mathbb{E}_\mu[w(X)] = \mu$ is obviously logconcave for all $\mu > 0$ and, then, the SBPD is underdispersed relative to the Poisson distribution (Theorem 1 but not Theorem 3). Since $w(0) = 0$ the SBPD does not have a punctual dual version (Definition 6). Its mean dual (if there exists) satisfies $\mathbb{E}_\mu[w^*(X)] = 1/\mu$ from (7), where w^* is the associated weight function defined on $\mathbb{N}$. Recall that the support of SBPD is $\mathbb{N}\backslash\{0\} = \{1, 2, \cdots\}$.

3.2 COM-Poisson distribution

The COM-Poisson distribution has been recently investigated by Shmueli *et al.* (2005) in the context of over- and underdispersion relative to the Poisson distribution. The name COM-Poisson is from the abbreviation of Conway and Maxwell (1962), which introduced this modified Poisson distribution connected with a queueing theory problem. The COM-Poisson distribution is defined by

$$\Pr(X = x|\gamma, \mu) = \frac{\mu^x}{(x!)^\gamma \sum_{k=0}^{\infty} \mu^k (k!)^{-\gamma}}, \quad x \in \mathbb{N}, \ \mu, \gamma > 0.$$

Extending γ to its two extremities, the COM-Poisson can thus be thought of as a continuous bridge between the geometric ($\gamma = 0$ with $0 < \mu < 1$), the

Poisson ($\gamma = 1$), and the Bernoulli ($\gamma = \infty$) distributions.

With respect to the Poisson ($\gamma = 1$), the COM-Poisson distribution is overdispersed when $\gamma \in [0, 1)$ and underdispersed when $\gamma \in (1, \infty]$. Indeed, the COM-Poisson distribution may also be viewed as a WPD with weight function

$$w(x; \gamma) = [\Gamma(x + 1)]^{1-\gamma}$$

does not depend on the (natural) parameter μ. Thus, since

$$\frac{d^2}{dx^2} \log w(x; \gamma) = (1 - \gamma) \sum_{k \in \mathbb{N}} \frac{1}{(k + x + 1)^2}$$

by properties of the gamma function (e.g. Abramowitz and Stegun, 1972, pages 258-259) we easily deduce the logconvexity (logconcavity) of $x \mapsto w(x; \gamma)$ when $\gamma \in [0, 1)$ ($\gamma \in (1, \infty]$), and then we apply Theorem 3 but not Theorem 1.

Furthermore, this family of COM-Poisson distributions is closed by the punctual duality for $\gamma \in [0, 2]$; that is, for a given COM-Poisson distribution with $\gamma_1 \in [0, 2]$ there exists another one with $\gamma_2 = 2 - \gamma_1 \in [0, 2]$ which is its punctual dual (Proposition 8). Note that for all underdispersed COM-Poisson distribution such that $\gamma > 2$, there does not exist its punctual dual because the corresponding WPD is not defined (Proposition 8 (i)).

3.3 Weights depending on natural parameter

Let us present two examples: first, we have

$$w(x; r, \mu) = \exp\{r|x - \mu|\} \tag{8}$$

for $r \in \mathbb{R}$ and $\mu > 0$ (Ridout and Besbeas, 2004) and, second, that is

$$w(x; r, \mu) = \left(\frac{x}{\mu}\right)^{rx} = \exp\{rI(x; \mu)\} \tag{9}$$

where $r \in \mathbb{R}$ and $I(x; \mu)$ is the Kullback-Leibler distance function between x and $\mu > 0$ (Efron, 1986).

For studying the over- and underdispersion situations of both WPD associated to (8) and to (9), we only use Theorem 3. Hence, from the trivial logconvexity (logconcavity) of $x \mapsto w(x; r, \mu)$ we have the closure of the corresponding families of WPD by punctual duality as follows: overdispersion relative to the Poisson when $r > 0$, underdispersion when $r < 0$, and equidispersion (or standard Poisson distribution) when $r = 0$.

For general expression of the weight function $w(x;\phi,\mu)$, a necessary and sufficient condition to examine the over- and underpersion relative to the Poisson distribution (and then the punctual duality) must depend on the structure of w and of the exponential model associated to the corresponding WPD. It is an open question.

4 Appendix: proofs of theorems

Proof of Theorem 1. Since the function

$$\mu \mapsto \mathbb{E}_\mu[w(X)] = \sum_{x\in\mathbb{N}} \frac{w(x)}{x!}\mu^x \exp(-\mu)$$

is $\mathcal{C}^\infty$ for all $\mu > 0$, the proof of Theorem 1 becomes trivial by using Lemma 2 and the following equivalences of assertions (i) and (ii) respectively as:
(i') $\frac{d^2}{d\mu^2} \log \mathbb{E}_\mu[w(X)]$ is positive (negative),
(ii') $Var(X^w)$ is greater (smaller) than $Var_\mu(X) = \mu = \mathbb{E}_\mu(X) = \mathbb{E}(X^w)$. □

Proof of Lemma 2. Let

$$p(x;\theta) = \frac{1}{x!}\exp\{\theta x - e^\theta\}, \quad x\in\mathbb{N},\ \theta = \log\mu \in \mathbb{R}$$

be the probability mass function of the original Poisson random variable X. From (1) the corresponding probability mass function of X^w is given by

$$p_w(x;\theta) = \frac{w(x)}{x!}\exp\{\theta x - e^\theta - \log \mathbb{E}_\theta[w(X)]\}, \quad x\in\mathbb{N}.$$

From (3) meaning that $w(x) = w(x;\phi)$ does not depend on the natural (or canonical) parameter $\theta = \theta(\mu)$, the probability mass function $p_w(x;\theta)$ is an element of natural exponential family for an eventual fixed ϕ (e.g. Jørgensen, 1997). It is characterized by its variance (function of θ):

$$Var(X^w) = \frac{d^2}{d\theta^2}\left(e^\theta + \log \mathbb{E}_\theta[w(X)]\right) = e^\theta + \frac{d^2}{d\theta^2}\log \mathbb{E}_\theta[w(X)].$$

Considering $\theta = \log\mu$ and (4), we easily deduce the expression of $Var(X^w)$ in terms of the mean $\mu > 0$:

$$Var(X^w) = \mu + \mu^2\,\frac{d^2}{d\mu^2}\log \mathbb{E}_\mu[w(X)].$$

The lemma is therefore proved. □

Proof of Theorem 3. Since $w(x) = w(x;\phi,\mu)$ depending on μ by (5), we compare straight $Var(X^w)$ to $Var_\mu(X) = \mu = \mathbb{E}_\mu(X) = \mathbb{E}(X^w)$ given by (4).

Indeed, from (1) the probability mass function of X^w is

$$p_w(x;\mu) = \frac{w(x)\mu^x e^{-\mu}}{x!\mathbb{E}_\mu[w(X)]}, \quad x \in \mathbb{N}, \ \mu > 0. \tag{10}$$

We successively obtain:

$$\begin{aligned}
\mathbb{E}[X^w] &= \frac{1}{\mathbb{E}_\mu[w(X)]} \sum_{x\geq 1} \frac{w(x)\mu^x e^{-\mu}}{(x-1)!} = \frac{\mu}{\mathbb{E}_\mu[w(X)]} \sum_{x\in\mathbb{N}} \frac{w(x+1)\mu^x e^{-\mu}}{x!} \\
&= \mu \ \frac{\mathbb{E}_\mu[w(X+1)]}{\mathbb{E}_\mu[w(X)]}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[(X^w)^2] &= \frac{1}{\mathbb{E}_\mu[w(X)]} \sum_{x\in\mathbb{N}} \frac{[x(x-1)+x]w(x)\mu^x e^{-\mu}}{x!} \\
&= \frac{\mu^2\mathbb{E}_\mu[w(X+2)] + \mu\mathbb{E}_\mu[w(X+1)]}{\mathbb{E}_\mu[w(X)]} \\
&= \mu^2 \ \frac{\mathbb{E}_\mu[w(X+2)]}{\mathbb{E}_\mu[w(X)]} + \mathbb{E}[X^w];
\end{aligned}$$

and, therefore, from (4) we can express

$$\begin{aligned}
Var(X^w) - \mu &= \mathbb{E}[(X^w)^2] - \mathbb{E}^2[X^w] - \mathbb{E}[X^w] \\
&= \frac{\mathbb{E}_\mu[w(X)]\mathbb{E}_\mu[w(X+2)] - \mathbb{E}_\mu^2[w(X+1)]}{\mu^{-2}\mathbb{E}_\mu^2[w(X)]}.
\end{aligned}$$

Since $\mu^{-2}\mathbb{E}_\mu^2[w(X)] > 0$ the sign of $Var(X^w) - \mu$ to get the over- and under-dispersion is provided by the one of its numerator, denoted by $N_\mu[w(X)]$ and which can be written as

$$\begin{aligned}
N_\mu[w(X)] &= \mathbb{E}_\mu[w(X)]\mathbb{E}_\mu[w(X+2)] - \mathbb{E}_\mu^2[w(X+1)] \\
&= \sum_x \frac{\mu^{2x}e^{-2\mu}}{(x!)^2}[w(x)w(x+2) - w^2(x+1)] \\
&\quad +2\sum_{x<y} \frac{\mu^{x+y}e^{-2\mu}}{x!y!} \left[w(x)w(y+2) - w(x+1)w(y+1)\right].
\end{aligned} \tag{11}$$

Now, the logconvexity of $w = \exp(h)$ implies

$$h(y+2) - h(y+1) > h(x+1) - h(x) \tag{12}$$

for all $x, y \in \mathbb{N}$ with $x \leq y$. This inequality (12) can be rewritten as

$$h(x) + h(y+2) > h(x+1) + h(y+1)$$

which is equivalent to

$$w(x)w(y+2) - w(x+1)w(y+1) > 0$$

for all $x, y \in \mathbb{N}$ with $x \leq y$. Thus, from (11) we deduce $N_\mu[w(X)] > 0$, that prove X^w is overdispersed relative to X.

If $w = \exp(h)$ is however logconcav, then the sense of the inequality (12) changes. We finally have $N_\mu[w(X)] < 0$ meaning X^w is underdispersed relative to X. This completes the proof. □

Proof of Proposition 5. From Gupta *et al.* (1997) and (10), a WPD is logconvex or logconcave according as

$$\begin{aligned} 0 \gtrless & \frac{p_w(x+1;\mu)}{p_w(x;\mu)} - \frac{p_w(x+2;\mu)}{p_w(x+1;\mu)} = \\ & \frac{\mu[(x+2)w^2(x+1) - (x+1)w(x)w(x+2)]}{(x+1)(x+2)w(x)w(x+1)} = \\ & \frac{\mu[w^2(x+1) - w(x)w(x+2)]}{(x+2)w(x)w(x+1)} + \frac{\mu w(x+1)}{(x+1)(x+2)w(x)}. \end{aligned}$$

Since $\mu > 0$, $w(x) > 0, \forall x \in \mathbb{N}$ and, similarly to the end of the proof of Theorem 3, the logconvexity or logconcavity of $x \mapsto w(x)$ can be expressed as

$$0 \gtrless w^2(x+1) - w(x)w(x+2),$$

the implication follows. □

Proof of Proposition 8. (i) Following Remark 7 (b) and the d'Alembert convergence criterion of positive series, the dual of X^w exists if there exists $c_0 \in [0, 1)$ such that

$$\frac{\Pr(X^{1/w} = x)}{\Pr(X^{1/w} = x-1)} = \frac{\mu\ w(x-1)}{x\ w(x)} \to c_0 \quad \text{as } x \to \infty.$$

(ii) Suppose $w_1 = \exp(h_1)$, where h_1 is convex (concave). Then $w_2 = \exp(-h_1)$ is logconcave (logconvex) and Proposition 3 allows to conclude the proof. □

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References

Abramowitz, M. and Stegun, I. A. (1972) *Handbook of Mathematical Functions.* New York: Dover.

Bosch, R. J. and Ryan, L. M. (1998) Generalized Poisson models arising from Markov processes. *Statist. Probab. Letters,* **39**, 205-212.

Cameron, A. C. and Johansson, P. (1997) Count data regression using series expansions: with applications. *J. Appl. Econometrics,* **12**, 203-223.

Castillo, J. and Pérez-Casany, M. (1998) Weighted Poisson distributions for overdispersion and underdispersion situations. *Ann. Inst. Statist. Math.,* **50**, 567-585

Castillo, J. and Pérez-Casany, M. (2005) Overdispersed and underdispersed Poisson generalizations. *J. Statist. Plann. Inference,* **134**, 486-500.

Conway, R. W. and Maxwell, W. L. (1962) A queueing model with state dependent service rates. *J. Industr. Engineering,* **12**, 132-136.

Dossou-Gbété, S., Kokonendji, C. C. and Mizère, D. (2005) Quelques tests de la loi de Poisson contre des alternatives générales basés sur l'indice de dispersion de Fisher. *Rev. Statist. Appl.* (to appear).

Efron, B. (1986) Double exponential families and their use in generalized linear regression. *J. Amer. Statist. Assoc.,* **81**, 709-721.

Fisher, R. A. (1934) The effects of methods of ascertainment upon the estimation of frequencies. *Ann. Eugenics,* **6**, 13-25.

Gelfand, A. E. and Dalal, S. R. A. (1990) A note on overdispersed exponential families. *Biometrika,* **77**, 55-64.

Gupta, P. L., Gupta, R. C. and Ong, S. H. (2004) Modelling count data by random effect Poisson model. *Sankhyā,* **66**, 548-565.

Gupta, P. L., Gupta, R. C. and Tripathi, R. C. (1997) On the monotone properties of discrete failure rates. *J. Statist. Plann. Inference,* **65**, 225-268.

Hougaard, P., Lee, M-L. T. and Whitmore, G. A. (1997) Analysis of overdispersed count data by mixtures of Poisson variables and Poisson processes. *Biometrics,* **53**, 1225-1238.

Jørgensen, B. (1997) *The Theory of Dispersion Models,* London: Chapman and Hall.

Kokonendji, C. C., Demétrio, C. G. B. and Dossou-Gbété, S. (2004) Some discrete exponential dispersion models: Poisson-Tweedie and Hinde-Demétrio classes. *SORT,* **28**, 201-214.

Luceño, A. (2005) Recursive characterization of a large family of discrete probability distribution showing extra-Poisson variation. *Statistics,* **39**, 261-267.

Patil, G. P. (2002) Weighted distributions. In *Encyclopedia of Environmetrics* (eds A. H. El-Shaarawi and W. W. Piegorsch), vol. 4, pp. 2369-2377. Chichester: John Wiley & Sons.

Patil, G. P. and Rao, C. R. (1978) Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics,* **34**, 179-184.

Puig, P. and Valero, J. (2005) Count data distributions: some characterizations with applications. *J. Amer. Statist. Assoc.* (to appear).

Rao, C. R. (1965) On discrete distributions arising out of methods of ascertainment. In *Classical and Contagious Discrete Distributions* (ed G. P. Patil), pp. 320-332. Calcutta: Pergamon Press and Statistical Publishing Society.

Ridout, M. S. and Besbeas, P. (2004) An empirical model for underdispersed count data. *Statist. Modelling,* **4**, 77-89.

Shmueli, G., Minka, T. P., Kadane, J. P., Borle, S. and Boatwright, P. (2005) A useful distribution for fitting discrete data: revival of the Conway-Maxwell-Poisson distribution. *J. Roy. Statist. Soc. Ser. C,* **54**, 127-142.

Steutel, F. W. (1985) Log-concave and log-convex distributions. In *Encyclopedia of Statistical Sciences* (eds S. Kotz, N. L. Johnson and C. B. Read), vol. 5, pp. 116-117. New York: Wiley.

Whalin, J. F. and Paris, J. (2002) A general family of overdispersed probability laws. *Belgian Actuar. Bull.,* **2**, 1-8.