Homogenization of a class of quasilinear elliptic equations in high-contrast fissured media

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Abstract

The aim of the paper is to study the asymptotic behavior of the solution of a quasilinear elliptic equation of the form

$$-\text{div} (a^\varepsilon(x)|\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon) + g(x)|u^\varepsilon|^{p-2}u^\varepsilon = S^\varepsilon(x) \quad \text{in} \quad \Omega,$$

with a high-contrast discontinuous coefficient $a^\varepsilon(x)$, where $\varepsilon$ is the parameter that characterizes the scale of the microstructure. The coefficient $a^\varepsilon(x)$ is assumed to degenerate everywhere in the domain $\Omega$ except in a thin connected microstructure of asymptotically small measure. It is shown that the asymptotical behavior of the solution $u^\varepsilon$ as $\varepsilon \to 0$ is described by a homogenized quasilinear equation with the coefficients calculated by local energetic characteristics of the domain $\Omega$.

1 Introduction

In this paper, we study the homogenization of the following quasilinear elliptic problem:

$$-\text{div} (a^\varepsilon(x)|\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon) + g(x)|u^\varepsilon|^{p-2}u^\varepsilon = S^\varepsilon(x) \quad \text{in} \quad \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $\varepsilon$ is a parameter tending to zero. We assume that $a^\varepsilon(x)$ does not degenerate along a thin connected microstructure $\Omega^\varepsilon_f$ (called

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fracture part) while in the complement to the fracture part $\Omega^\epsilon_m$ (called matrix part) $a^\epsilon(x)$ is a positive function vanishing asymptotically as $\epsilon \to 0$. The rate of degeneration will be specified later.

The basic set of equations considered here arises, for example, from compressible flows in porous media, non-Newtonian flow etc. through thin fissures. This problem is closely related to the so-called double porosity homogenization models widely discussed in the mathematical literature (see, e.g., [16]). The linear double porosity model was first studied in [4], and was then revisited in the mathematical literature by many other authors (see, e.g., [16] for a review). Nonlinear models were treated in [10, 21]. Then a general non-periodic model and a random model were considered in [7, 6], respectively. Let us also mention that the homogenization of nonlinear elliptic equations has been a problem of many years and a number of methods have been developed. There is an extensive literature on this subject. We will not attempt a literature review here, but merely mention a few references, see for instance [9, 11, 12, 14, 20], and the references therein.

In contrast with the above mentioned works, where the measure of the fracture part remains uniformly positive, in our model we will assume that the measure of the fracture part is asymptotically small. This problem, in the linear case, was considered in [1, 2, 5, 23], the singular double porosity model was studied in [8].

In the present paper we deal with a quasilinear elliptic problem in a domain with asymptotically small fissure part. Following the approach introduced in [17], instead of a classical periodicity assumption, we impose certain conditions on the so-called local energetic characteristics associated with the boundary value problem (1.1). These characteristics include a penalization term. Following the scheme developed in [1, 23], we obtain the homogenization result by combining the local characteristics method with an appropriate extension condition from the fracture part to the whole domain $\Omega$.

Since the measure of the fracture part is small we cannot use the usual notions of convergence and compactness. Instead we introduce the convergence and compactness adapted to the singularity of the fracture measure.

The homogenized equation takes the form:

$$-\partial_x a_i(x, \nabla u) + B(x)|u|^{p-2}u = \rho(x)S(x) \quad \text{in} \quad \Omega,$$

where the functions $a_i \ (i = 1, 2, \ldots, n)$ and $B(x)$ are defined in terms of the above mentioned local characteristics.

The paper is organized as follows. In Section 2 all necessary mathematical notations are defined, the microscopic problem is formulated and the general assumptions are stated. In Section 3 we introduce the notions of convergence and compactness in domains of asymptotically degenerating measure. The main result of the paper is formulated in Section 4. It will then proved in Sections 5 and 6. Finally, in Section 7 we present 2D and 3D periodic examples. In these examples the coefficients of the homogenized problem are calculated explicitly.
2 Statement of the problem and assumptions

In this Section, we describe a microscopic model for a quasilinear elliptic equation in high-contrast fissured media. Let \( \Omega = \Omega_f^\varepsilon \cup \Omega_m^\varepsilon \) be a bounded domain in \( \mathbb{R}^n \) (\( n = 2, 3 \)) with piecewise smooth boundary \( \partial \Omega \), and let

\[
\text{meas} \Omega_f^\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{2.1}
\]

Moreover, throughout the paper we assume that the set \( \Omega_f^\varepsilon \) is asymptotically distributed in a regular way in the domain \( \Omega \), i.e., there exists a constant \( C > 0 \) such that for any ball \( V(y, r) = \{ x : |x - y| < r \} \) of radius \( r \) centered at \( y \in \Omega \) and \( \varepsilon > 0 \) small enough \( (\varepsilon \leq \varepsilon_0(r)) \),

\[
C^{-1} r^n \geq \mu^\varepsilon \text{meas} (\Omega_f^\varepsilon \cap V(x, r)) \geq Cr^n, \tag{2.2}
\]

where

\[
\mu^\varepsilon = \frac{\text{meas} \Omega}{\text{meas} \Omega_f^\varepsilon}. \tag{2.3}
\]

We consider the following variational problem:

\[
\mu^\varepsilon \int_{\Omega} \{ a^\varepsilon(x)|\nabla u|^p + g(x)|u|^p - p S^\varepsilon(x) u^\varepsilon \} \, dx \to \inf, \quad u^\varepsilon \in W^{1,p}(\Omega), \tag{2.4}
\]

where \( p > 1 \), \( g \) is a smooth positive function in \( \overline{\Omega} \) such that \( g(x) \geq C > 0 \);

\[
S^\varepsilon(x) = 1_f^\varepsilon(x) S(x) \tag{2.5}
\]

with \( S \in L^{p'}(\Omega) \), where \( p' \) satisfies \((1/p) + (1/p') = 1\) and \( 1_f^\varepsilon = 1_f^\varepsilon(x) \) being the characteristic function of the set \( \Omega_f^\varepsilon \); \( a^\varepsilon = a^\varepsilon(x) \) is a measurable function such that:

\[
0 < a_0 \leq a^\varepsilon(x) \leq a_0^{-1} \quad \text{in} \quad \Omega_f^\varepsilon; \tag{2.6}
\]

\[
0 < a_1(\varepsilon) \leq a^\varepsilon(x) \leq a_2(\varepsilon) \quad \text{in} \quad \Omega_m^\varepsilon \tag{2.7}
\]

with

\[
\mu^\varepsilon a_2(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{2.8}
\]

It is known (see, e.g., [18]) that, for any \( \varepsilon > 0 \), there exists a unique solution \( u^\varepsilon \in W^{1,p}(\Omega) \) of the variational problem (2.4), and that \( u^\varepsilon \) solves the Neumann boundary value problem for the corresponding Euler equation:

\[
-\text{div} (a^\varepsilon(x)|\nabla u|^p - 2\nabla u^\varepsilon) + g(x)|u|^p - 2u^\varepsilon = S^\varepsilon(x) \quad \text{in} \quad \Omega. \]

3 Convergence in domains of degenerating measure

Due to the vanishing measure of the fissure part, we should define the convergence of sequences according to the singularity of the fissure measure. Assume that the family of domains \( \Omega_f^\varepsilon \subset \Omega \) (\( \varepsilon > 0 \)) satisfies (2.1) and (2.2). In this Section, following [23] (see
also [19] or [25] for similar considerations), we introduce the concept of convergence in domains \( \Omega^\varepsilon_j \) as \( \varepsilon \to 0 \).

We adopt the following notation: \( \| \cdot \|_\Omega \) and \( \| \cdot \|_{1,\Omega} \) are the norms in the spaces \( L^p(\Omega) \) and \( W^{1,p}(\Omega) \) (\( 1 < p < +\infty \)), respectively; \( \text{Lip}(M, \Omega) \) is the class of continuous functions \( u \) in \( \Omega \) such that \( |u(x)| \leq M \) and \( |u(x) - u(y)| \leq M|x - y| \) for any \( x, y \in \Omega \).

**Definition 3.1** A sequence of functions \( \{u^\varepsilon \in L^p(\Omega^\varepsilon_j)\} \) is said to \( D^p_{\Omega^\varepsilon_j} \)-converge to a function \( u \in L^p(\Omega) \) if there exists an approximating sequence \( \{u_M \in \text{Lip}(M, \Omega), M = 1, 2, \ldots\} \) converging strongly in \( L^p(\Omega) \) to \( u \) as \( M \to \infty \), and

\[
\lim_{M \to \infty} \lim_{\varepsilon \to 0} \frac{1}{\text{meas} \Omega^\varepsilon_j} \|u^\varepsilon - u_M\|_{\Omega^\varepsilon_j}^p = 0. \tag{3.1}
\]

Notice that under condition (2.2) the limiting function \( u \) in Definition 3.1 is independent of approximating sequence \( \{u_M, M = 1, 2, \ldots\} \).

**Remark 1** If in Definition 3.1 \( u \) is a smooth function in \( \Omega \) then (3.1) could be rewritten as follows:

\[
\lim_{\varepsilon \to 0} \frac{1}{\text{meas} \Omega^\varepsilon_j} \|u^\varepsilon - u\|_{\Omega^\varepsilon_j}^p = 0. \tag{3.2}
\]

In a natural way one defines the notion of compactness with respect to the \( D^p_{\Omega^\varepsilon_j} \)-convergence.

**Definition 3.2** A sequence \( \{u^\varepsilon \in L^p(\Omega^\varepsilon_j)\} \) is a \( D^p_{\Omega^\varepsilon_j} \)-compact set if from any its subsequence one can extract a \( D^p_{\Omega^\varepsilon_j} \)-convergent subsequence.

In what follows we mainly deal with sequences of functions \( u^\varepsilon \in W^{1,p}(\Omega^\varepsilon_j) \) such that

\[
\|u^\varepsilon\|_{1,p,\Omega^\varepsilon_j}^p \leq C \text{meas} \Omega^\varepsilon_j. \tag{3.3}
\]

From now on, \( C \) is a generic constant independent of \( \varepsilon \). Furthermore, in this paper we restrict ourselves to domains \( \Omega^\varepsilon_j \) satisfying the so-called "strong connectedness" condition (the SC-condition).

**Definition 3.3** A family of domains \( \Omega^\varepsilon_j \) is said to satisfy the SC-condition if for any sequence \( \{u^\varepsilon \in C^1(\Omega^\varepsilon_j)\} \) satisfying (3.3), and any \( M \) (\( M = 1, 2, \ldots \)), there exists a family of subsets \( \mathcal{G}_M \subset \Omega^\varepsilon_j \) such that \( u^\varepsilon \in \text{Lip}(M, \Omega^\varepsilon_j \setminus \mathcal{G}_M) \) and

\[
\text{meas} \mathcal{G}_M^\varepsilon = \frac{\phi(M)}{M^p} \text{meas} \Omega^\varepsilon_j, \quad \|u^\varepsilon\|_{\mathcal{G}_M}^p = \phi(M) \text{meas} \Omega^\varepsilon_j
\]

for all \( \varepsilon, \varepsilon \leq \varepsilon_0(M) \), where \( \lim_{M \to \infty} \phi(M) = 0 \).

Now we formulate a sufficient condition for \( D^p_{\Omega^\varepsilon_j} \)-compactness in the class of domains \( \Omega^\varepsilon_j \) satisfying (2.2) and the SC-condition. The following Theorem holds (see [25, 3]).
where \( p > \frac{1}{n} \), we impose certain conditions on local energy characteristics of the subdomains \( \Omega^\varepsilon \). For this, we assume that \( \Omega^\varepsilon \) is a family of domains satisfying the SC-condition. Then any sequence \( \{u^\varepsilon \in W^{1,p}(\Omega^\varepsilon)\} \) satisfying (3.3) is a \( D^p_{\Omega^\varepsilon} \)-compact set.

Remark 2 In the proof of Theorem 3.4 we construct the sequence \( \{u^\varepsilon\} \) such that \( u^\varepsilon(x) = u^\varepsilon_\Omega(x) \) for \( x \in \Omega^\varepsilon_\Omega \), where \( \Omega^\varepsilon_\Omega = \Omega^\varepsilon \setminus \mathcal{G}^\varepsilon_\Omega \). The functions \( u^\varepsilon_\Omega \) satisfy the Lipschitz condition and it follows from Witney’s theorem (see [24]) that these functions can be extended from \( \Omega^\varepsilon_\Omega \) to the whole \( \Omega \). This means that the SC-condition could be formulated as an extension condition for the functions defined in the domain \( \Omega^\varepsilon_\Omega \) to the domain \( \Omega \) with some distortion on the set \( \mathcal{G}^\varepsilon_\Omega \) whose measure is small with respect to the measure of the set \( \Omega^\varepsilon_\Omega \).

4 Formulation of the main result

In this Section we introduce local energy characteristics of the sets \( \Omega^\varepsilon_\Omega \) and \( \Omega^\varepsilon_m \) associated to the variational problem (2.4), and formulate the main result of the paper. We study the asymptotic behavior of \( u^\varepsilon \) solutions of the variational problem (2.4) as \( \varepsilon \to 0 \). The classical periodicity assumption is here substituted by an abstract one covering a variety of concrete behaviors such as the periodicity, the almost periodicity, and many more besides. For this, we assume that \( \Omega^\varepsilon_\Omega \subset \Omega \) is a disperse medium, i.e. the following assumptions hold:

\begin{align}
\text{(C.1) there exists a continuous function } & \rho(x) > 0 \text{ in } \bar{\Omega} \text{ such that }
\lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon h^{-n} \text{meas} [\Omega^\varepsilon_\Omega \cap K^\varepsilon_h] = \rho(x),
\end{align}

for any open cube \( K^\varepsilon_h \) centered at \( x \in \Omega \) with lengths equal to \( h > 0 \);

\begin{align}
\text{(C.2) the family of domains } & \Omega^\varepsilon_\Omega \text{ (} \varepsilon > 0 \text{) satisfies the SC-condition (see Definition 3.3).}
\end{align}

Instead of the classical periodicity assumption on the microstructure of the disperse media, we impose certain conditions on local energy characteristics of the subdomains \( \Omega^\varepsilon_\Omega \) and \( \Omega^\varepsilon_m \).

For \( z \in \Omega \) we define:
- the functional associated to the energy in \( \Omega^\varepsilon_\Omega \):
\begin{align}
E_{\varepsilon,h}(z; \vec{q}) &= \inf_{u^\varepsilon} \mu^\varepsilon \int_{K^\varepsilon_h \cap \Omega^\varepsilon_\Omega} \left\{ \alpha^\varepsilon(x)|\nabla u^\varepsilon|^p + h^{-p-\gamma}|u^\varepsilon - (x - z, \vec{q})|^p \right\} \, dx,
\end{align}

where \( p > \gamma > 0 \), \( \vec{q} = \{q_1, q_2, \ldots, q_n\} \in \mathbb{R}^n \), and where the infimum is taken over \( u^\varepsilon \in W^{1,p}(K^\varepsilon_h \cap \Omega^\varepsilon_\Omega) \);
- the functional associated to the exchange between the matrix and the fissure system:
\begin{align}
b_{\varepsilon,h}(z) &= \inf_{u^\varepsilon} \mu^\varepsilon \int_{K^\varepsilon_h} \left\{ \alpha^\varepsilon(x)|\nabla u^\varepsilon|^p + g(x)1^\varepsilon_m(x)|w^\varepsilon|^p + h^{-p-\gamma}1^\varepsilon_f(x)|w^\varepsilon - 1|^p \right\} \, dx,
\end{align}

where \( 1^\varepsilon_m = 1^\varepsilon_m(x) \), \( 1^\varepsilon_f = 1^\varepsilon_f(x) \) are the characteristic functions of the sets \( \Omega^\varepsilon_m \) and \( \Omega^\varepsilon_f \), respectively, and where the infimum is taken over \( w^\varepsilon \in W^{1,p}(K^\varepsilon_h) \).
Our further assumptions are the following:

\[(C.3)\text{ for any } x \in \Omega \text{ there exist the limits}\]

\[
\lim_{h \to 0} \lim_{\varepsilon \to 0} h^{-n} E^\varepsilon h(x; \bar{q}) = \lim_{h \to 0} \lim_{\varepsilon \to 0} h^{-n} E^\varepsilon h(x; \bar{q}) = A(x, \bar{q})
\]

with the function \(A(x, \bar{q})\) such that \(A(x, \cdot) \in C^{2+\beta}(\mathbb{R}^n), \beta > 0 \) and \(A(\cdot, \bar{q}) \in C(\Omega)\); moreover,

\[
C_1 |\bar{q}|^{p-2} |\xi|^2 \geq A_{p, p_j} \xi \xi_j \geq C_2 |\bar{q}|^{p-2} |\xi|^2 \quad (C_1, C_2 > 0);
\]

\[
A(x, \bar{q}) \geq C_3 (|\bar{q}|^{p} - 1) \quad (C_3 > 0);
\]

\[(C.4)\text{ for any } x \in \Omega \text{ there exist the limits}\]

\[
\lim_{h \to 0} \lim_{\varepsilon \to 0} h^{-n} b^\varepsilon h(x) = \lim_{h \to 0} \lim_{\varepsilon \to 0} h^{-n} b^\varepsilon h(x) = b(x),
\]

where \(b \in C(\Omega)\).

Now we are in position to formulate the main result of this paper.

**Theorem 4.1** Let conditions \((C.1)\)–\((C.4)\) be satisfied. Then the solution \(u^\varepsilon\) of problem (2.4) \(D^p_{\Omega^\varepsilon}\)-converges to \(u\) the solution of the problem:

\[
J_{hom}(u) \equiv \int_{\Omega} \{ A(x, \nabla u) + B(x)|u|^p - pp(x)S(x)u \} dx \to \inf \quad u \in W^{1,p}(\Omega),
\]

where \(B(x) = g(x)\rho(x) + b(x)\).

### 5 Preliminary results

In this Section we construct a convenient approximation for the solution of the variational problem (2.4) in the subdomains \(\Omega_m^\varepsilon, \Omega_f^\varepsilon \subset \Omega\). To this end we introduce first the following notation.

Let \(\{x^\alpha\}\) be a periodic grid in \(\Omega\) with a period \(h' := h - h^{1+\gamma/p}, (\varepsilon \ll h \ll 1)\). Let us cover the domain \(\Omega\) by the cubes \(K^\alpha_h\) of length \(h > 0\) centered at the points \(x^\alpha\) and \(x\). We associate with this covering a partition of unity \(\{\varphi^\alpha\} : 0 \leq \varphi^\alpha(x) \leq 1; \varphi^\alpha(x) = 0 \text{ for } x \notin K^\alpha_h; \varphi^\alpha(x) = 1 \text{ for } x \in K^\alpha_h \setminus \cup_{\beta \neq \alpha} K^\beta_h; \sum^\alpha \varphi^\alpha(x) = 1 \text{ for } x \in \Omega; |\nabla \varphi^\alpha(x)| \leq C h^{-1-\gamma/p}\).

Denote by \(K^\alpha_{h'}\) and \(\Pi^\alpha_{h}\) the cube of length \(h'\) centered at the point \(x^\alpha\), and the set \(K^\alpha_h \setminus K^\alpha_{h'}\), respectively.

**Lemma 5.1** Assume that conditions \((C.1), (C.4)\) are satisfied. Then for each \(h > 0\) there exist sets \(B^\varepsilon_h \subset \Omega^\varepsilon_f\) and functions \(Y^\varepsilon_h\) such that

(i) \(0 \leq Y^\varepsilon_h(x) \leq 1\) in \(\Omega\);
(ii) \(Y^\varepsilon_h(x) = 1\) in \(\Omega^\varepsilon_f \setminus B^\varepsilon_h\);
(iii) \(\lim_{\varepsilon \to 0} \mu^\varepsilon \text{ meas } B^\varepsilon_h = O(h^\gamma)\) as \(h \to 0\);
(iv) for any function \( w \in C^1(\Omega) \), we have
\[
\lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega} \{a^\varepsilon(x) |\nabla Y_h^\varepsilon|^p + g(x)|Y_h^\varepsilon|^p \} |w|^p \, dx \leq \int_{\Omega} B(x)|w|^p \, dx + o(1) \quad (h \to 0). \tag{5.1}
\]

Proof of Lemma 5.1. Let \( w_h^{\varepsilon,\alpha} \) be a minimizer of the functional in (4.2) with \( z = x^\alpha \).

It follows from condition (C.4) that, as \( h \to 0 \),
\[
\lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega_h} \{a^\varepsilon(x) |\nabla w_h^{\varepsilon,\alpha}|^p + g(x) \mathbf{1}_m(x)|w_h^{\varepsilon,\alpha}|^p \} \, dx = o(h^n), \tag{5.2}
\]
\[
\lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega_h} \mathbf{1}_f(x)|w_h^{\varepsilon,\alpha} - 1|^p \, dx = o(h^{n+p+\gamma}). \tag{5.3}
\]

In addition, using conditions (C.1) and (C.4), we obtain
\[
\lim_{\varepsilon \to 0} \mu^\varepsilon \int_{K_h^\varepsilon} \{(a^\varepsilon(x) |\nabla w_h^{\varepsilon,\alpha}|^p + g(x)|w_h^{\varepsilon,\alpha}|^p \} \, dx \leq h^n B(x^\alpha) + o(h^n) \tag{5.4}
\]
as \( h \to 0 \). Besides, since \( w_h^{\varepsilon,\alpha} \) minimizes the functional in (4.2) we have \( 0 \leq w_h^{\varepsilon,\alpha}(x) \leq 1 \), and
\[
\lim_{\varepsilon \to 0} \mu^\varepsilon \operatorname{meas} B_h^{\varepsilon,\alpha} \leq C h^{n+\gamma}, \tag{5.5}
\]
where \( B_h^{\varepsilon,\alpha} = \{ x \in K_h^\varepsilon \cap \Omega_f^\varepsilon : w_h^{\varepsilon,\alpha}(x) \leq 1 - h \} \). The inequality (5.5) means that the measure of the set \( B_h^{\varepsilon,\alpha} \), where the function \( w_h^{\varepsilon,\alpha} \) minimizing the functional in (4.2) is not close to 1, is small with respect to the measure of the set \( \Omega_f^\varepsilon \cap K_h^\varepsilon \).

Let us introduce the function:
\[
W_h^{\varepsilon,\alpha} = \begin{cases} 
1, & \text{if } w_h^{\varepsilon,\alpha} \geq 1 - h; \\
(1 - h)^{-1} w_h^{\varepsilon,\alpha}, & \text{otherwise.}
\end{cases} \tag{5.6}
\]

It is clear that \( |W_h^{\varepsilon,\alpha} - 1| \leq |w_h^{\varepsilon,\alpha} - 1| \). One can easily show that the function \( W_h^{\varepsilon,\alpha} \) satisfy the estimates (5.2)–(5.4). We set
\[
B_h^{\varepsilon} = \bigcup_{\alpha} B_h^{\varepsilon,\alpha}, \quad Y_h^{\varepsilon}(x) = \sum_{\alpha} W_h^{\varepsilon,\alpha}(x) \varphi_\alpha(x).
\]

Then, using the properties of the functions \( W_h^{\varepsilon,\alpha} \) and \( \{ \varphi_\alpha \} \) and taking into account the estimate (5.5), it is easy to show that the functions \( Y_h^{\varepsilon}(x) \) and the sets \( B_h^{\varepsilon} \) satisfy conditions (i)–(iv) of Lemma 5.1. This completes the proof of Lemma 5.1.

Lemma 5.2 Let conditions (C.2), (C.3) be satisfied and let \( w \) be a smooth function in \( \Omega \). Then for any \( h > 0 \) and \( M \in \{1, 2, \ldots\} \), there exist functions \( W_{Mh}^\varepsilon \in W^{1,p}(\Omega) \), such that
(i) \( |W_{Mh}^\varepsilon(x) - w(x)| \leq CMh \quad \text{in} \ \Omega; \)
(ii) \( |W_{Mh}^\varepsilon(x) - W_{Mh}^\varepsilon(y)| \leq CM|x - y| \quad \text{for any} \ x, y \in \Omega; \)
(iii) the inequality holds:
\[
\lim_{M \to \infty} \lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega_f} a^\varepsilon(x) |\nabla W_{Mh}^\varepsilon|^p \, dx \leq \int_{\Omega} A(x, \nabla w) \, dx.
\]
Proof of Lemma 5.2. Let \( w \) be a smooth function in \( \Omega \) and let \( v^\varepsilon_h \) be a minimizer of the functional in (4.1) for \( z = x^\alpha \) and \( \bar{q} = \nabla w(x^\alpha) \). Since \( v^\varepsilon_h \) minimizes the functional in (4.1), we have
\[
\sup_{x \in K^\varepsilon_h \cap \Omega^\varepsilon_f} |v^\varepsilon_h(x)| \leq \frac{h}{2}.
\]
(5.7)

Besides, by virtue of condition (C.3), for \( \varepsilon \) small enough (\( \varepsilon \leq \tilde{\varepsilon}(h) \)), we have
\[
\mu^\varepsilon \int_{\Pi^\varepsilon_h \cap \Omega^\varepsilon_f} |\nabla v^\varepsilon_h|^p \, dx = o(h^n),
\]
(5.8)
\[
\mu^\varepsilon \int_{\Pi^\varepsilon_h \cap \Omega^\varepsilon_f} |v^\varepsilon_h - (x - x^\alpha, \nabla w(x^\alpha))|^p \, dx = o(h^{n+p+\gamma}),
\]
(5.9)
\[
\mu^\varepsilon \int_{K^\varepsilon_h \cap \Omega^\varepsilon_f} a^\varepsilon(x) |\nabla v^\varepsilon_h|^p \, dx \leq h^n A(x^\alpha, \nabla w(x^\alpha)) + o(h^n)
\]
(5.10)
as \( h \to 0 \). The estimates (5.8)--(5.10) are uniform with respect to \( x^\alpha \) on any compact subset of \( \Omega \).

Let us consider the function
\[
w^\varepsilon_h(x) = \sum_{\alpha} \{w(x) + v^\varepsilon_h(x) - (x - x^\alpha, \nabla w(x^\alpha))\} \varphi_{\alpha}(x).
\]
(5.11)

Then, using (5.8)--(5.10) and the properties of the functions \( \varphi_{\alpha} \), we obtain
\[
\mu^\varepsilon \int_{\Omega^\varepsilon_f} a^\varepsilon(x)|\nabla w^\varepsilon_h|^p \, dx \leq \int_{\Omega} A(x, \nabla w) \, dx + o(1)
\]
(5.12)
as \( \varepsilon \leq \tilde{\varepsilon}(h) \) and \( h \to 0 \). Besides, according to (5.7) and (5.11), we have
\[
\sup_{x \in \Omega^\varepsilon_f} |w^\varepsilon_h(x) - w(x)| \leq Ch.
\]
(5.13)

Since the domains \( \Omega^\varepsilon_f \) satisfy the SC–condition (see Definition 3.3), for any \( M = 1, 2, \ldots, \) there exist sets \( Q^\varepsilon_{Mh} \) and functions \( W^\varepsilon_{Mh} \in \text{Lip}(M, \Omega) \) such that \( W^\varepsilon_{Mh} = w^\varepsilon_h \) in \( \Omega^\varepsilon_f \setminus Q^\varepsilon_{Mh} \), and
\[
\lim_{M \to \infty} M^p \lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon \text{meas } Q^\varepsilon_{Mh} = 0.
\]
(5.14)

Now the statements of the Lemma follow from (5.12)--(5.13) and the properties of the functions \( W^\varepsilon_{Mh} \). Lemma 5.2 is proved. \( \square \)

Lemma 5.3 Let conditions (C.1) and (C.4) be satisfied. Then, for any function \( w \in W^{1,p}(\Omega) \), there exists a sequence \( \{w^\varepsilon \in C^{1}(\Omega)\} \) which \( D^{p}_{\Omega_f} \)-converges to \( w \) and such that
\[
\lim_{\varepsilon \to 0} I^\varepsilon[w^\varepsilon] = \lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega} \{a^\varepsilon(x)|\nabla w^\varepsilon|^p + g(x)|w^\varepsilon|^p\} \, dx \leq C_0 \|w\|^p_{1,\Omega},
\]
(5.15)
where \( C_0 > 0 \) is independent of \( w \).
Proof of Lemma 5.3. The proof of the Lemma will be given in two steps.

Step 1. Let us first assume that \( w \in C^\infty(\Omega) \). Then we construct a sequence \( \{ w^\varepsilon \} \) as follows. Consider the function \( w^\varepsilon_h(x) = w(x) Y_h^\varepsilon(x) \), where \( Y_h^\varepsilon(x) \) is defined in Lemma 5.1. According to Lemma 5.1, there are sets \( B_h^\varepsilon \subset \Omega_j^\varepsilon \) with \( \operatorname{meas} B_h^\varepsilon \leq Ch^\gamma \operatorname{meas} \Omega_j^\varepsilon \) (here \( \varepsilon \) is small enough, \( \varepsilon \leq \varepsilon(h) \)), such that \( w^\varepsilon_h = w \) in \( \Omega_j^\varepsilon \setminus B_h^\varepsilon \) and

\[
\int_{B_h^\varepsilon} |w^\varepsilon_h - w|^p \, dx \leq \int_{\Omega_j^\varepsilon} |w|^p \, dx. \tag{5.16}
\]

Consider now the integral \( I^\varepsilon[w^\varepsilon_h] \) (cf. (5.15)). Using (2.6) it is easy to show that

\[
I^\varepsilon[w^\varepsilon_h] \leq a_0^{-1} \mu \int_{\Omega_j^\varepsilon} |\nabla w|^p \, dx + 2^{p-1} \mu \int_{\Omega} \{ a^\varepsilon(x) |\nabla Y_h^\varepsilon|^p + g(x) |Y_h^\varepsilon|^p \} |w|^p \, dx +
\]

\[
+ 2^{p-1} \mu \int_{B_h^\varepsilon \cup \Omega_m^\varepsilon} a^\varepsilon(x) |Y_h^\varepsilon|^p |\nabla w|^p \, dx. \tag{5.17}
\]

Moreover, it follows from condition (2.7) and Lemma 5.1 that

\[
\lim_{\varepsilon \to 0} \mu \int_{B_h^\varepsilon \cup \Omega_m^\varepsilon} a^\varepsilon(x) |Y_h^\varepsilon|^p |\nabla w|^p \, dx = O(h^\gamma)
\]

as \( h \to 0 \). On the other hand, conditions (C.1) and (2.2) imply the convergence

\[
\mu \int_{\Omega_j^\varepsilon} |\nabla w|^p \, dx \to \int_{\Omega} \rho(x) |\nabla w|^p \, dx, \quad \text{as} \quad \varepsilon \to 0. \tag{5.18}
\]

Define a sequence \( \{ \varepsilon_j \} \), \( \varepsilon_j \downarrow 0 \), such that for each \( h_j = 1/j \), it holds \( \varepsilon_j \leq \varepsilon(h_j) \) and set

\[
w^\varepsilon = w^\varepsilon_h \bigg|_{h=1/j} \quad \text{when} \quad \varepsilon_j \geq \varepsilon > \varepsilon_{j+1}.
\]

It follows from (5.17), (5.18), and the assertion (iv) of Lemma 5.1 that \( \{ w^\varepsilon \} \) satisfies (5.15) with \( C_0 = \max\{ a_0^{-1} \max_{x \in \Omega} \rho(x), 2^{m-1} \max_{x \in \Omega} B(x) \} \). Also from the definition of the function \( w^\varepsilon \) and (5.16) we get

\[
\lim_{\varepsilon \to 0} \frac{1}{\operatorname{meas} \Omega_j^\varepsilon} \int_{\Omega_j^\varepsilon} |w^\varepsilon - w|^p \, dx = 0. \tag{5.19}
\]

This means that the sequence \( \{ w^\varepsilon \} \) \( D_{\Omega_j^\varepsilon}^p \)-converges to the function \( w \) (see Remark 1).

Clearly, the functions \( w^\varepsilon \) can be approximated by smooth ones in such a way that for the approximation sequence (5.15) and (5.19) hold true.

Step 2. Now consider an arbitrary function \( w \in W^{1,p}(\Omega) \), and approximate \( w \) by smooth functions \( w_M \in \operatorname{Lip}(M, \Omega) \) \((M = 1, 2, \ldots)\) such that

\[
\| w - w_M \|_{1, \Omega} \leq \frac{1}{k(M)} \quad \text{with} \quad k(M) \to +\infty \quad \text{as} \quad M \to +\infty. \tag{5.20}
\]
According to Step 1, there is a sequence \( \{w^\varepsilon_M\} \) such that
\[
\frac{1}{\text{meas } \Omega^\varepsilon_f \Omega_f} \int |w^\varepsilon_M - w_M|^p \, dx \leq \frac{1}{M}; \tag{5.21}
\]
and
\[
I^\varepsilon[w^\varepsilon_M] \leq C_1 \|w\|_{1, \Omega}^p,
\]
where \( \varepsilon \) is sufficiently small, \( \varepsilon \leq \delta(M) \), \( C_1 \) does not depend on \( \varepsilon, M \). Here \( \delta(M) \to 0 \) as \( M \to \infty \). Moreover, in view of (2.2), \( \delta(M) \) can be chosen in such a way that, as \( \varepsilon \leq \delta(M) \),
\[
\frac{1}{\text{meas } \Omega^\varepsilon_f \Omega_f} \int |w_M - w_L|^p \, dx \leq C_2 \|w_M - w_L\|_{\Omega}^p, \quad \text{for all } L = 1, \ldots, M - 1, \tag{5.22}
\]
with a constant \( C_2 \) independent of \( \varepsilon, M \). We choose a sequence \( \{\delta_j\}_{j=1,2, \ldots} \), \( \delta_j \downarrow 0 \), such that \( \delta_j \leq \delta(j) \) and set
\[
w^\varepsilon = w^\varepsilon_M \quad \text{for } \varepsilon \in [\delta_{M+1}, \delta_M].
\]
It is easy to see that the sequence \( \{w^\varepsilon\} \) satisfies (5.15). Moreover,
\[
\frac{1}{\text{meas } \Omega^\varepsilon_f \Omega_f} \int |w^\varepsilon - w_{M_0}|^p \, dx \leq \frac{2}{\text{meas } \Omega^\varepsilon_f \Omega_f} \int |w_M^\varepsilon - w_M|^p \, dx + 2C_2 \|w_M - w_{M_0}\|_{\Omega}^p
\]
for any \( M_0 \) and \( \varepsilon \in [\delta_{M+1}, \delta_M] \). Therefore, according to (5.20)–(5.22), the sequence \( \{w^\varepsilon\} \) \( D^p_{\Omega^\varepsilon_f} \)-converges to \( w \). Lemma 5.3 is proved. \( \square \)

**Lemma 5.4** Let conditions (C.1)–(C.3) be satisfied, and let \( \{v^\varepsilon \in C^1(\Omega^\varepsilon_f)\} \) be a sequence satisfying (3.3). Then there exists a family of continuous functions \( \{v^\varepsilon_M\} \ (M = 1, 2, \ldots) \) in \( \Omega \) such that
(i) \( \lim_{M \to \infty} \lim_{\varepsilon \to 0} \mu^\varepsilon \text{meas } \{x \in \Omega^\varepsilon_f : v^\varepsilon(x) \neq v^\varepsilon_M(x)\} = 0; \)
(ii) \( v^\varepsilon_M \in \text{Lip}(CM, \Omega) \) with a constant \( C > 0 \) independent of \( \varepsilon \) and \( M \);
(iii) for any \( M \), there exists a subsequence \( \{v^\varepsilon_{M_j}\} \ (\varepsilon_j \to 0) \) converging uniformly in \( \Omega \) to a function \( v_M \in \text{Lip}(CM, \Omega); \)
(iv) for any sequence of sets \( Q^\varepsilon_M \subset \Omega^\varepsilon_f \) such that
\[
\lim_{M \to \infty} M^p \lim_{\varepsilon \to 0} \mu^\varepsilon \text{meas } Q^\varepsilon_M = 0, \tag{5.23}
\]
we have
\[
\lim_{\varepsilon=\varepsilon_j \to 0} \left\{ \mu^\varepsilon \int_{\Omega^\varepsilon_f \setminus Q^\varepsilon_M} a^\varepsilon(x) |\nabla v^\varepsilon| \, dx - \int_{\Omega} A(x, \nabla v_M) \, dx \right\} = o(1), \tag{5.24}
\]
as \( M \to \infty \). Moreover,
\[
\lim_{M \to \infty} \lim_{\varepsilon \to 0} \mu^\varepsilon \|v^\varepsilon\|^p_{Q^\varepsilon_M} = 0. \tag{5.25}
\]
Proof of Lemma 5.4. Using (C.2) we have that there is a set $G^\varepsilon_M$ such that $\mu^\varepsilon \text{meas } G^\varepsilon_M = M^{-p}\phi(M)$ and $\mu^\varepsilon\|v^\varepsilon\|_{L^p(G^\varepsilon_M)} = \phi(M)$ for $\varepsilon \leq \varepsilon_0(M)$ with $\phi(M) \to 0$ as $M \to \infty$ and $v^\varepsilon \in \text{Lip}(M, \Omega^\varepsilon_f \setminus G^\varepsilon_M)$. This implies that (5.25) holds true. Moreover, according to Witney’s theorem (see [24]), the functions $v^\varepsilon \in \text{Lip}(CM, \Omega)$ and $v^\varepsilon = v^\varepsilon_M$ in $\Omega^\varepsilon_f \setminus G^\varepsilon_M$, where $C > 0$ is independent of $\varepsilon$ and $M$. Then, for any fixed $M$, $\{v^\varepsilon_M\}$ is a compact set in $C(\Omega)$. Therefore, there is a subsequence $\{v^\varepsilon_j\}$ ($\varepsilon_j \to 0$) converging uniformly in $\Omega$ to a function $v_M \in \text{Lip}(CM, \Omega)$. Thus, the family $\{v^\varepsilon_M\}$ satisfies assertions (i)–(iii) of Lemma 5.4.

It remains to prove the assertion (iv). Let $v_{M\delta}$ be a smooth function in $\Omega$ such that

$$
\|v_{M\delta} - v_M\|_{1,\Omega} < \delta. \tag{5.26}
$$

We want to construct a sequence $\{v^\varepsilon_{M\delta}\}$ satisfying

$$
\lim_{\varepsilon \to 0} \mu^\varepsilon\|v^\varepsilon_{M\delta}\|_{L^p(\Omega^\varepsilon_f)}^p \leq C\delta^p, \quad \text{and} \quad \lim_{\varepsilon \to 0} \mu^\varepsilon\|v^\varepsilon_{M\delta} - (v_{M\delta} - v_M)\|_{L^p(\Omega^\varepsilon_f)}^p = 0. \tag{5.27}
$$

For this we introduce a sequence of smooth functions $w_k$ such that, for any $k = 1, 2, \ldots$

$$
\int_\Omega |\nabla w_k|^p \, dx \leq \delta^p + \frac{1}{k} \quad \text{and} \quad \int_\Omega |w_k - (v_{M\delta} - v_M)|^p \, dx \leq \frac{1}{k}.
$$

Then, since $w_k$, $|\nabla w_k|$, $v_{M\delta}$ and $v_M$ are continuous functions in $\Omega$, we obtain from (2.2) and condition (C.1),

$$
\lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega^\varepsilon_f} |\nabla w_k|^p \, dx = \int_\Omega |\nabla w_k|^p \rho(x) \, dx,
$$

$$
\lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega^\varepsilon_f} |w_k - (v_{M\delta} - v_M)|^p \, dx = \int_\Omega |w_k - (v_{M\delta} - v_M)|^p \rho(x) \, dx.
$$

This implies that there exists a sequence $\{\varepsilon_k\}$, $\varepsilon_k \downarrow 0$, such that

$$
\mu^\varepsilon \int_{\Omega^\varepsilon_f} |\nabla w_k|^p \, dx \leq C \left(\delta^p + \frac{1}{k}\right), \quad \mu^\varepsilon \int_{\Omega^\varepsilon_f} |w_k - (v_{M\delta} - v_M)|^p \, dx \leq C \frac{1}{k},
$$

as $\varepsilon < \varepsilon_k$. For $\varepsilon_{k+1} \leq \varepsilon < \varepsilon_k$ we set $v^\varepsilon_{M\delta} = w_k$. Then the sequence $\{v^\varepsilon_{M\delta}\}$ satisfies (5.27).

In order to prove (5.24) we first notice that

$$
\int_{\Omega^\varepsilon_f} a^\varepsilon(x)|\nabla v^\varepsilon_M|^p \, dx = \int_{\Omega^\varepsilon_f} a^\varepsilon(x)|\nabla (v^\varepsilon_M + v^\varepsilon_{M\delta})|^p \, dx +
$$

$$
+ \int_{\Omega^\varepsilon_f} a^\varepsilon(x) \left\{|\nabla v^\varepsilon_M|^p - |\nabla (v^\varepsilon_M + v^\varepsilon_{M\delta})|^p\right\} \, dx.
$$

Then, according to (5.27) we have

$$
\lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega^\varepsilon_f} a^\varepsilon(x)|\nabla v^\varepsilon_M|^p \, dx \geq \lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega^\varepsilon_f} a^\varepsilon(x)|\nabla (v^\varepsilon_M + v^\varepsilon_{M\delta})|^p \, dx + \xi_1(\delta), \tag{5.28}
$$

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where \( \xi_1(\delta) \to 0 \) as \( \delta \to 0 \).

Now we cover the domain \( \Omega \) by cubes \( K^\alpha_h = \{ x \in \Omega : |x_l - y_l^\alpha| \leq h/2 \} \) with non-intersecting interiors. Consider an arbitrary cube \( K^\alpha_h \) such that \( K^\alpha_h \cap \partial \Omega = \emptyset \). We set

\[ \psi^\varepsilon_M(x) = v_M^\varepsilon(x) + v_{M\delta}^\varepsilon(x) - v_{M\delta}(y^\alpha). \]

The sequence \( \{ v_{M\varepsilon}^\varepsilon \} \) converges uniformly in \( \Omega \) to \( v_M \) as \( \varepsilon \to 0 \). Then (5.27) implies that

\[ \int_{\Omega^\varepsilon \cap K_h^\alpha} a^\varepsilon(x)|\nabla \psi^\varepsilon_{M\delta}|^p \, dx = \]

\[ = \int_{\Omega^\varepsilon \cap K_h^\alpha} \left\{ a^\varepsilon(x)|\nabla \psi^\varepsilon_{M\delta}|^p + h^{-p-\gamma} |\psi^\varepsilon_{M\delta} - (\nabla v_{M\delta}(y^\alpha), x - y^\alpha)|^p \right\} \, dx - \xi_2^\alpha(\varepsilon, h, M) \]

with \( \lim_{\varepsilon = \varepsilon_j \to 0} \xi_2^\alpha(\varepsilon, h, M) = O(h^{n+p-\gamma}) \) as \( h \to 0 \). Therefore, it follows from condition (C.3) that

\[ \lim_{\varepsilon = \varepsilon_j \to 0} \mu^\varepsilon \int_{\Omega^\varepsilon} a^\varepsilon(x)|\nabla (v_M^\varepsilon + v_{M\delta}^\varepsilon)|^p \, dx = \lim_{\varepsilon = \varepsilon_j \to 0} \mu^\varepsilon \int_{\Omega^\varepsilon} a^\varepsilon(x)|\nabla \psi^\varepsilon_{M\delta}|^p \, dx \geq \]

\[ \geq \int_\Omega A(x, \nabla v_{M\delta}) \, dx. \] (5.29)

Now, passing to the limit in (5.29) as \( \delta \to 0 \) and using (5.26), (5.28) we obtain

\[ \lim_{\varepsilon = \varepsilon_j \to 0} \mu^\varepsilon \int_{\Omega^\varepsilon} a^\varepsilon(x)|\nabla v_M^\varepsilon|^p \, dx \geq \int_\Omega A(x, \nabla v_M) \, dx. \] (5.30)

Finally, it is easy to see that

\[ \lim_{\varepsilon = \varepsilon_j \to 0} \mu^\varepsilon \int_{\Omega^\varepsilon \setminus \mathcal{Q}^\varepsilon_M} a^\varepsilon(x)|\nabla v_M^\varepsilon|^p \, dx \geq \lim_{\varepsilon = \varepsilon_j \to 0} \mu^\varepsilon \int_{\Omega^\varepsilon} a^\varepsilon(x)|\nabla v_M^\varepsilon|^p \, dx - \xi(M), \] (5.31)

where

\[ \xi(M) = \lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\mathcal{Q}^\varepsilon_{M \cup Q}^\varepsilon} a^\varepsilon(x)|\nabla v_M^\varepsilon|^p \, dx = o(1) \quad \text{as} \quad M \to \infty. \]

Assertion (iv) of Lemma 5.4 follows from (5.30) and (5.31). This completes the proof of Lemma 5.4.

\[ \square \]

6 Proof of Theorem 4.1

We begin this Section by obtaining a priori estimates for the minimizer of problem (2.4):

\[ J^\varepsilon[u^\varepsilon] \equiv \mu^\varepsilon \int_\Omega \{ a^\varepsilon(x)||\nabla u^\varepsilon|^p + g(x)|u^\varepsilon|^p - p S^\varepsilon(x) u^\varepsilon \} \, dx \to \inf, \quad u^\varepsilon \in W^{1,p}(\Omega), \] (6.1)
Since $J^\varepsilon[u^\varepsilon] \leq J^\varepsilon[0] = 0$, by virtue of the Young inequality and (2.5) we have
\[
\mu^\varepsilon \int_\Omega \{a^\varepsilon(x)|\nabla u^\varepsilon|^p + g(x)|u^\varepsilon|^p\} \, dx \leq C_1 \mu^\varepsilon \|S\|_{\Omega^p}^p \leq C_2,
\]
where the constants $C_1, C_2$ do not depend on $\varepsilon$. Then it follows from (6.2) that
\[
\mu^\varepsilon \|u^\varepsilon\|_{p,\Omega_j}^p \leq C.
\]
Hence, $\{u^\varepsilon\}$ is a $D^p_{\Omega_j}$-compact set and one can extract a subsequence (still denoted by $\{u^\varepsilon\}$) $D^p_{\Omega_j}$-converging to a function $u \in L^p(\Omega)$. Let us show that $u = u(x)$ is a solution of the variational problem (4.5). This will be done in two steps.

### 6.1 Step 1. Upper bound
Let $w = w(x)$ be an arbitrary smooth function in $\Omega$ and let $Y^\varepsilon_h$, $W^\varepsilon_{Mh}$, $B^\varepsilon_h$ be the same as in Lemmas 5.1 and 5.2. We set
\[
\vartheta^\varepsilon_{Mh}(x) = Y^\varepsilon_h(x)W^\varepsilon_{Mh}(x).
\]
It is clear that $\vartheta^\varepsilon_{Mh} \in W^{1,p}(\Omega)$.

First we prove that
\[
\lim_{M \to \infty} \lim_{h \to 0} \lim_{\varepsilon \to 0} J^\varepsilon[\vartheta^\varepsilon_{Mh}] \leq J_{\text{hom}}[w],
\]
where
\[
J_{\text{hom}}[w] = \int_\Omega \{A(x, \nabla w) + B(x)|w|^p - p\rho(x)S(x)w\} \, dx
\]
with $B(x) = (g\rho + b)(x)$. We have:
\[
J^\varepsilon[\vartheta^\varepsilon_{Mh}] = \mu^\varepsilon \int_\Omega \{a^\varepsilon(x)|\nabla \vartheta^\varepsilon_{Mh}|^p + g(x)|\vartheta^\varepsilon_{Mh}|^p - p S^\varepsilon(x)\vartheta^\varepsilon_{Mh}\} \, dx.
\]
Consider the third term in (6.6). It follows from (C.1), (2.5), assertions (i), (ii) of Lemma 5.1, and assertion (i) of Lemma 5.2 that
\[
\lim_{M \to \infty} \lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon \int_\Omega S^\varepsilon(x)\vartheta^\varepsilon_{Mh}(x) \, dx = \int_\Omega S(x)w(x)\rho(x) \, dx.
\]
Consider the second term in (6.6). We have
\[
\mu^\varepsilon \int_\Omega g(x)|\vartheta^\varepsilon_{Mh}|^p \, dx = \mu^\varepsilon \int_\Omega g(x)|Y^\varepsilon_h|^p|w|^p \, dx + \mu^\varepsilon \int_\Omega g(x)|Y^\varepsilon_h|^p \{|W^\varepsilon_{Mh}|^p - |w|^p\} \, dx.
\]
By the assertions (iv) of Lemma 5.1 and (i) of Lemma 5.2 we get
\[
\lim_{M \to \infty} \lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon \int_\Omega g(x)|Y^\varepsilon_h|^p \{|W^\varepsilon_{Mh}|^p - |w|^p\} \, dx = 0.
\]
For the first term in (6.6) we have

\[
\mu^\varepsilon \int_{\Omega} a^\varepsilon(x)|\nabla \varphi|^p \, dx = \mu^\varepsilon \int_{\Omega_m} a^\varepsilon(x)|\nabla W_{Mh}^\varepsilon|^p \, dx + \\
+ \mu^\varepsilon \int_{B_h^\varepsilon a^\varepsilon(x)|\nabla Y_{h1}^\varepsilon W_{Mh}^\varepsilon + \nabla W_{Mh}^\varepsilon Y_{h1}^\varepsilon|^p \, dx + \mu^\varepsilon \int_{\Omega_m} a^\varepsilon(x)|\nabla Y_{h1}^\varepsilon W_{Mh}^\varepsilon + \nabla W_{Mh}^\varepsilon Y_{h1}^\varepsilon|^p \, dx.
\] (6.10)

The assertion (iii) of Lemma 5.2 implies that the following limit relation holds true:

\[
\lim_{M \to \infty} \lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega_m} a^\varepsilon(x)|\nabla W_{Mh}^\varepsilon|^p \, dx \leq \int_{\Omega} A(x, \nabla w) \, dx.
\] (6.11)

It is clear that

\[
\mu^\varepsilon \int_{B_h^\varepsilon} a^\varepsilon|\nabla Y_{h1}^\varepsilon W_{Mh}^\varepsilon + \nabla W_{Mh}^\varepsilon Y_{h1}^\varepsilon|^p \, dx = \mu^\varepsilon \int_{B_h^\varepsilon} a^\varepsilon|\nabla Y_{h1}^\varepsilon|^p \, dw + \\
+ \mu^\varepsilon \int_{B_h^\varepsilon} a^\varepsilon|\nabla Y_{h1}^\varepsilon|^p \, \{W_{Mh}^\varepsilon - |w|^p\} \, dx + \mu^\varepsilon \int_{B_h^\varepsilon} a^\varepsilon \{\nabla \varphi_M^\varepsilon |W_{Mh}^\varepsilon|^p - |\nabla Y_{h1}^\varepsilon W_{Mh}^\varepsilon|^p\} \, dx.
\] (6.12)

The statements (iv) of Lemma 5.1 and (i) of Lemma 5.2 imply

\[
\lim_{M \to \infty} \lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon \int_{B_h^\varepsilon} a^\varepsilon(x)|\nabla Y_{h1}^\varepsilon|^p \{W_{Mh}^\varepsilon - |w|^p\} \, dx = 0.
\] (6.13)

The assertions (i), (iii), (iv) of Lemma 5.1 and (i), (ii) of Lemma 5.2 imply

\[
\lim_{M \to \infty} \lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon \int_{B_h^\varepsilon} a^\varepsilon(x) \{\nabla \varphi_M^\varepsilon |W_{Mh}^\varepsilon|^p - |\nabla Y_{h1}^\varepsilon W_{Mh}^\varepsilon|^p\} \, dx = 0.
\] (6.14)

For the third term in the right-hand side of (6.10) we have

\[
\mu^\varepsilon \int_{\Omega_m} a^\varepsilon|\nabla Y_{h1}^\varepsilon W_{Mh}^\varepsilon + \nabla W_{Mh}^\varepsilon Y_{h1}^\varepsilon|^p \, dx = \mu^\varepsilon \int_{\Omega_m} a^\varepsilon|\nabla Y_{h1}^\varepsilon|^p \, dw + \\
+ \mu^\varepsilon \int_{\Omega_m} a^\varepsilon|\nabla Y_{h1}^\varepsilon|^p \{W_{Mh}^\varepsilon - |w|^p\} \, dx + \mu^\varepsilon \int_{\Omega_m} a^\varepsilon \{\nabla \varphi_M^\varepsilon |W_{Mh}^\varepsilon|^p - |\nabla Y_{h1}^\varepsilon W_{Mh}^\varepsilon|^p\} \, dx
\] (6.15)

and by the condition (2.7), the assertions (i), (iv) of Lemma 5.1, and (i), (ii) of Lemma 5.2 we get

\[
\lim_{M \to \infty} \lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega_m} a^\varepsilon(x)|\nabla Y_{h1}^\varepsilon|^p \{W_{Mh}^\varepsilon - |w|^p\} \, dx = 0
\] (6.16)

and

\[
\lim_{M \to \infty} \lim_{h \to 0} \lim_{\varepsilon \to 0} \mu^\varepsilon \int_{\Omega_m} a^\varepsilon(x) \{\nabla \varphi_M^\varepsilon |W_{Mh}^\varepsilon|^p - |\nabla Y_{h1}^\varepsilon W_{Mh}^\varepsilon|^p\} \, dx = 0.
\] (6.17)

Now we obtain (6.4) from (6.7)–(6.17) and assertion (iv) of Lemma 5.1. Since \(u^\varepsilon\) minimizes the functional \(J^\varepsilon\), it follows from (6.4) that

\[
\lim_{\varepsilon \to 0} J^\varepsilon[u^\varepsilon] \leq J_{\text{hom}}[w]
\] (6.18)

for any smooth function \(w\). By density arguments, (6.18) holds for any function \(w \in W^{1,p}(\Omega)\) as well.
6.2 Step 2. Lower bound

Let \( \{u^\varepsilon\} \) be a sequence of solutions of the variational problem (6.1) which \( D_{\Omega_f}^p \) -converges to a function \( u \). We want to show that

\[
\lim_{\varepsilon \to 0} J^\varepsilon[u^\varepsilon] \geq J_{\text{hom}}[u].
\] (6.19)

According to Lemma 5.4 there exists a family of functions \( \{u_M^\varepsilon\} \) \( \varepsilon \in (0, 1) \) such that

\[
\lim_{M \to \infty} \lim_{\varepsilon \to 0} \mu^\varepsilon \|u^\varepsilon - u_M^\varepsilon\|_{\Omega_f}^p = 0.
\] (6.20)

Moreover, for any \( M \), there is a subsequence \( \{u_{M_j}^\varepsilon\} \) (\( \varepsilon_j \to 0 \)) converging uniformly in \( \Omega \) to a function \( u_M \), and

\[
\lim_{\varepsilon = \varepsilon_j \to 0} \mu^\varepsilon \int_{\Omega_f} a^\varepsilon(x)|\nabla u^\varepsilon|^p \, dx \geq \lim_{M \to \infty} \int_{\Omega} A(x, \nabla u_M) \, dx - \varepsilon(M),
\] (6.21)

where \( \varepsilon(M) \to 0 \) as \( M \to \infty \). Since the sequence \( \{u^\varepsilon\} \) \( D_{\Omega_f}^p \) -converges to \( u \) and \( \{u_{M_j}^\varepsilon\} \) converges uniformly to \( u_M \), (6.20) implies that the functions \( u_M \) converge in \( L^p(\Omega) \) to \( u \) as \( M \to \infty \). In addition, it follows from (6.21) and (4.4) that the sequence \( \{u_M\} \) is bounded in \( W^{1,p}(\Omega) \). Thus, \( u \in W^{1,p}(\Omega) \).

Let us approximate \( u \) by smooth functions \( u_\delta(x) \) (\( \delta > 0 \)) in \( \Omega \),

\[
\|u_\delta - u\|_{1,\Omega} \leq \delta,
\] (6.22)

and set \( w_\delta(x) = u_\delta(x) - u(x) \). By virtue of Lemma 5.3 there exists a sequence \( \{w_\delta^\varepsilon\} \) which \( D_{\Omega_f}^p \) -converges to \( w_\delta \) and satisfies the bound

\[
\lim_{\varepsilon \to 0} I^\varepsilon[w_\delta^\varepsilon] \leq C\delta,
\] (6.23)

where \( C \) does not depend on \( u_\delta \), and the functional \( I^\varepsilon \) is defined by (5.15). We set

\[
u_\delta^\varepsilon = w_\delta^\varepsilon + u^\varepsilon.
\] (6.24)

The sequence \( \{u_\delta^\varepsilon\} \) \( D_{\Omega_f}^p \) -converges to \( u_\delta \) and according to (6.23),

\[
\lim_{\varepsilon \to 0} J^\varepsilon[u_\delta^\varepsilon] \leq \lim_{\varepsilon \to 0} J^\varepsilon[u^\varepsilon] + \varepsilon(\delta),
\] (6.25)

where \( \varepsilon(\delta) \to 0 \) as \( \delta \to 0 \) (by passing, if necessary, to a subsequence we can assume that the limit in the right-hand side of (6.25) exists).

Since \( u_\delta(x) \) is a smooth function, from Remark 1 we deduce

\[
\lim_{\varepsilon \to 0} \mu^\varepsilon \|u_\delta^\varepsilon - u_\delta\|_{\Omega_f}^p = 0.
\] (6.26)

Inequality (6.25) and Lemma 5.4 imply the existence of functions \( u_{\delta M}^\varepsilon \in \text{Lip}(CM, \Omega) \) (\( M = 1, 2, \ldots \)) and sets \( Q_M^\varepsilon \) such that \( u_{\delta M}^\varepsilon(x) = u_\delta^\varepsilon(x) \) for \( x \in \Omega_f^\varepsilon \setminus Q_M^\varepsilon \) and \( \mu^\varepsilon \text{meas} Q_M^\varepsilon \).
uniformly in to a function for any \( v \) as \( \varepsilon \to 0 \).

Clearly, it follows from (6.26) that there exists a sequence \( \{r^\varepsilon\} \), \( r^\varepsilon \to 0 \), and sets \( B^\varepsilon_M \) such that

\[
\lim_{\varepsilon \to 0} \mu^n \text{meas } B^\varepsilon_M = 0 \quad \text{and} \quad |u^\varepsilon_M(x) - u_\delta(x)| \leq r^\varepsilon \quad \text{in } \Omega^\varepsilon_f \setminus Z^\varepsilon_M,
\]

where \( Z^\varepsilon_M = Q^\varepsilon_M \cup B^\varepsilon_M \). Let us define the functions

\[
v^\varepsilon_M(x) = \begin{cases} 
    u_\delta(x) + r^\varepsilon, & \text{if } u^\varepsilon_M(x) > u_\delta(x) + r^\varepsilon; \\
    u^\varepsilon_M(x), & \text{if } |u^\varepsilon_M(x) - u_\delta(x)| \leq r^\varepsilon; \\
    u_\delta(x) - r^\varepsilon, & \text{if } u^\varepsilon_M(x) < u_\delta(x) - r^\varepsilon.
\end{cases}
\]

Clearly, \( v^\varepsilon_M \in \text{Lip}(CM, \Omega) \). Moreover, the functions \( v^\varepsilon_M \) converge uniformly in \( \Omega \) to \( u_\delta \) as \( \varepsilon \to 0 \).

We set \( V^\varepsilon_M = u^\varepsilon_M - v^\varepsilon_M \) and consider the left-hand side of the inequality (6.25). Since \( v^\varepsilon_M(x) = u^\varepsilon_M(x) \) for \( x \in \Omega^\varepsilon_f \setminus Z^\varepsilon_M \), we have

\[
I^\varepsilon[u^\varepsilon_M] = \mu^n \left( \int_{\Omega^\varepsilon_f} a^\varepsilon(x) |\nabla V^\varepsilon_M|^p \, dx + \int_{\Omega^n_m} \{a^\varepsilon(x) |\nabla u^\varepsilon_M|^p + g(x)|u^\varepsilon_M|^p \} \, dx \right) + \\
\quad + \mu^n \left( \int_{\Omega^\varepsilon_f} a^\varepsilon(x) |\nabla u^\varepsilon_M|^p \, dx + \int_{\Omega^\varepsilon_f} g(x)|u^\varepsilon_M|^p \, dx \right) + \\
\quad + \mu^n \left( \int_{\Omega^\varepsilon_f} a^\varepsilon(x) |\nabla (u^\varepsilon_M - v^\varepsilon_M)|^p \, dx - \int_{\Omega^\varepsilon_f} a^\varepsilon(x) |\nabla (u_\delta - v^\varepsilon_M)|^p \, dx \right) \equiv \theta_1 + \theta_2 + \theta_3.
\]

Consider the first term on the right-hand side of (6.28). First we define \( \Omega_\zeta \subset \Omega \):

\[
\Omega_\zeta = \{ x \in \Omega : |u_\delta(x)| > 2\zeta \},
\]

where \( \zeta > 0 \). Let us cover \( \Omega_\zeta \) by cubes \( K^\alpha_h \) of length \( h \) centered at \( x^\alpha \) with nonintersecting interiors. For \( \varepsilon \) and \( h \) sufficiently small, we have \( |v^\varepsilon_M| > \zeta \) in \( K^\alpha_h \). One can show that for \( x \in \Omega^\varepsilon_f \cap K^\alpha_h \) we have:

\[
\left( 1 + A_1 h^{\frac{\alpha - 1}{p-1}} \right) a^\varepsilon(x) |\nabla (u_\delta - v^\varepsilon_M)|^p \geq
\]

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where $A_1, A_2$ are positive constants independent of $\varepsilon, \delta, M$. In a similar way, for $x \in \Omega^c_\varepsilon \cap K^\alpha_h$ we have:

\[
(1 + A_1 h^{p-1}) a^{\varepsilon}(x) |\nabla u^\varepsilon_\delta|^p \geq a^{\varepsilon}(x) |\nabla u^\varepsilon_\delta|^p - a_2(\varepsilon) A_2 \left(1 + \frac{1}{h^p}\right) |u^\varepsilon_\delta|^p |\nabla v^\varepsilon_{\delta M}|^p, \tag{6.30}
\]

where $a_2(\varepsilon)$ is defined in (2.7).

Now we make use of (6.26) and of the definition of the function $v^\varepsilon_{\delta M}(x)$ and its properties. For any $K^\alpha_h \subset \Omega^c_\varepsilon$, we obtain:

\[
\mu^\varepsilon \int_{K^\alpha_h \cap \Omega^c_\varepsilon} a^{\varepsilon}(x) |\nabla V^\varepsilon_{\delta M}|^p dx + \mu^\varepsilon \int_{K^\alpha_h \cap \Omega^c_\varepsilon} \{a^{\varepsilon}(x) |\nabla u^\varepsilon_\delta|^p + g(x) |u^\varepsilon_\delta|^p\} dx \geq 0.
\]

for $\varepsilon$ small enough ($\varepsilon \leq \varepsilon(h)$) and $h \to 0$. Condition (C.1) implies

\[
\lim_{\varepsilon \to 0} \mu^\varepsilon \left\{ \int_{K^\alpha_h} a^{\varepsilon}(x) |\nabla \left( \frac{u^\varepsilon_\delta}{v^\varepsilon_{\delta M}} \right)|^p dx + \int_{K^\alpha_h \cap \Omega^c_\varepsilon} g(x) \left| \frac{u^\varepsilon_\delta}{v^\varepsilon_{\delta M}} \right|^p dx \right\} \geq h^n b(x^\alpha) + o(h^n) \tag{6.32}
\]

as $h \to 0$. Now it follows from (6.31) and (6.32) that

\[
\lim_{\varepsilon \to 0} \theta^1 \geq \int_{\Omega^c_\varepsilon} b(x) |u^\varepsilon_\delta|^p dx. \tag{6.33}
\]

Taking into account the definition of $\Omega^c_\varepsilon$ and passing to the limit as $\varepsilon \to 0$ in (6.33) we get

\[
\lim_{\varepsilon \to 0} \theta^1 \geq \int_{\Omega} b(x) |u^\varepsilon_\delta|^p dx. \tag{6.34}
\]

In order to estimate $\theta_2$ from below in (6.28) we argue as follows. Using Lemma 5.4, (6.26) and condition (C.1), we obtain

\[
\lim_{\varepsilon \to 0} \theta_2 \geq \int_{\Omega} A(x, \nabla u^\varepsilon_\delta) dx + \int_{\Omega} g(x) |u^\varepsilon_\delta|^p \rho(x) dx + o(1) \tag{6.35}
\]

as $M \to \infty$. Since the first term on the right-hand side of (6.35) is a weakly lower semicontinuous functional in $W^{1,p}(\Omega)$, and functions $u^\varepsilon_\delta M$ converge in $L^p(\Omega)$ to $u^\varepsilon_\delta$ as $M \to \infty$, we have

\[
\lim_{M \to \infty} \lim_{\varepsilon \to 0} \theta_2 \geq \int_{\Omega} \{A(x, \nabla u^\varepsilon_\delta) + g(x) \rho(x) |u^\varepsilon_\delta|^p\} dx. \tag{6.36}
\]
Finally, we consider the third term on the right-hand side of (6.28). Using (6.27) and (2.6) we get

$$|\theta_3| \leq C_1 \mu^\varepsilon \int_{\Omega^\varepsilon_f \cap \mathcal{Z}_M^\varepsilon} |\nabla u_\delta| \left\{ |\nabla u_\delta^{\varepsilon-1}| + |\nabla u_\delta|^{p-1} \right\} \, dx,$$

where $C_1$ is a constant independent of $\varepsilon, \delta, M$. Since $u_\delta(x)$ is a smooth function in $\Omega$, we finally get

$$|\theta_3| \leq C_2 \mu^\varepsilon \int_{\Omega^\varepsilon_f \cap \mathcal{Z}_M^\varepsilon} \left\{ 1 + |\nabla u_\delta|^{p-1} \right\} \, dx,$$

(6.37)

where $C_2$ is a constant independent of $\varepsilon, M$. Now it is easy to see that the definition of the function $u_\delta$, (6.24), (6.3), (6.23), the estimate for the measure of $\mathcal{Z}_M^\varepsilon$, and Hölder’s inequality yield

$$\lim_{M \to \infty} \lim_{\varepsilon \to 0} |\theta_3| = 0.$$

(6.38)

Thus it follows from (6.34), (6.36), (6.38) (2.5), and condition (C.1) that

$$\lim_{\varepsilon \to 0} J^\varepsilon[u_\delta] \geq J_{hom}[u_\delta].$$

(6.39)

This inequality, (6.22), and (6.25) immediately yield (6.19).

Inequalities (6.18), (6.19) mean that if a subsequence of solutions of problem (6.1) $D^\varepsilon_{\Omega_f}$–converges to a function $u = u(x)$, then $u$ minimizes the functional $J_{hom}$ in $W^{1,p}(\Omega)$, i.e. $u$ is a solution of the problem (4.5). Since $b(x) \geq 0$, this problem has a unique solution and the whole sequence of solutions of problem (6.1) $D^\varepsilon_{\Omega_f}$–converges to the function $u$. This completes the proof of Theorem 4.1.

\[ \square \]

7 Periodic examples

As an application of the previous general result, we give now two examples of fissured media, where the distribution of the fracture part is specified.

Theorem 4.1 of Section 4 provides sufficient conditions for the existence of the homogenized problem (4.5). The goal of this Section is to prove that for appropriate periodic examples all the conditions of Theorem 4.1 are satisfied and to compute the coefficients of the homogenized problem (4.5) explicitly. We will study the following variational problem:

$$\mu^\varepsilon \int_{\Omega} \{ a^\varepsilon(x)|\nabla u^\varepsilon|^{p} + g|u^\varepsilon|^{p} - p S^\varepsilon(x) u^\varepsilon \} \, dx \to \inf, \quad u^\varepsilon \in W^{1,p}(\Omega),$$

(7.1)

where $p \geq 2$ and

$$a^\varepsilon(x) = \alpha_f^\varepsilon 1_f^\varepsilon(x) + \alpha_m^\varepsilon 1_m^\varepsilon(x); \quad S^\varepsilon(x) = 1_f^\varepsilon(x) S(x)$$

(7.2)

with $S \in L^p(\Omega)$, $g, \alpha_f, \alpha_m$ are strictly positive constants and $\theta > 0$ is a parameter.

In the following subsections we study a periodic thin connected microstructure $\Omega_f^\varepsilon$ of two different types.
7.1 2D periodic example

Let \( \Omega = \Omega_\varepsilon \cup \Omega_m \) be a bounded domain in \( \mathbb{R}^2 \) with piecewise smooth boundary \( \partial \Omega \). We define the set \( \Omega_\varepsilon \) as follows. Let \( \mathcal{P}^\varepsilon \subset \mathbb{R}^2 \) be the simplest lattice structure consisting of two \( \varepsilon \)-periodic systems of thin strips oriented in the coordinate directions. The width of the strips is equal to \( d \varepsilon \),

\[
d\varepsilon = d\varepsilon^{\theta/p} \quad (d > 0, \theta > p \geq 2).
\]

This case describes the critical thickness of the fissures when the exchange process between the matrix and the fissures is not negligible. We set \( \Omega_\varepsilon = \Omega \cap \mathcal{P}^\varepsilon \). Then \( \Omega_\varepsilon \) is made of periodically (with the period \( \varepsilon \)) distributed squares \( M_i^\varepsilon \) with centers at \( x_{i,\varepsilon} \).

Let us formulate the homogenization result for this example.

**Theorem 7.1** Let \( \{u^\varepsilon\} \) be the sequence of solutions of problem (7.1)–(7.2). Then \( \{u^\varepsilon\} \) \( D_{\Omega_\varepsilon} \)-converges to \( u \) the solution of the problem:

\[
\int_{\Omega} \left\{ \frac{\alpha_f}{2} (|u_{x_1}|^p + |u_{x_2}|^p) + B|u|^p - pS(x)u \right\} \, dx \to \inf \quad u \in W^{1,p}(\Omega),
\]

where

\[
B = g + \frac{2(\alpha_m)\frac{1}{2}}{d} \left( \frac{g}{p - 1} \right)^{\frac{p-1}{p}}.
\]

7.1.1 Proof of Theorem 7.1

We have to verify conditions (C.1)–(C.4) of Section 4 and to calculate the functions \( \rho(x) \), \( A(x, \bar{q}) \), and \( b(x) \). For this example, the main difficulty is the verification of condition (C.4).

First it is easy to see that \( \text{meas} \Omega_\varepsilon = 2d\varepsilon^{-1}\text{meas} \Omega + o(1) \) as \( \varepsilon \to 0 \) and consequently

\[
\mu^\varepsilon = \frac{\varepsilon}{2d\varepsilon} + o(1) \quad (\varepsilon \to 0).
\]

Now let \( K_h^\varepsilon \) be an open square with length \( h \) \((0 < \varepsilon \ll h < 1)\) centered at \( z \in \Omega \).

First we check condition (C.1). Since \( \text{meas}(K_h^\varepsilon \cap \Omega_\varepsilon) \sim 2d\varepsilon^{-1} h^2 \), it is clear that condition (C.1) is satisfied and

\[
\rho(x) = 1.
\]

The fact that the family of domains \( \{\Omega_\varepsilon\} \) satisfies condition (C.2) \((SC\text{-condition})\) is known from [25] (see also [19]).

Concerning condition (C.3), it was already considered in [22] and [3] in a more general situation. Applying the results of [22] we get

\[
A(x, \bar{q}) = \frac{\alpha_f}{2} (|q_1|^p + |q_2|^p).
\]
It remains to check condition (C.4). Denote by $\mathcal{M}$ the unit square in the space $\mathbb{R}^2$, $\mathcal{M} = \{x \in \mathbb{R}^2 : |x_i| < 1/2\}$. Consider the following boundary value problem:

$$
\begin{align*}
\Delta_p W^\varepsilon + \beta^\varepsilon |W^\varepsilon|^{p-2} W^\varepsilon &= 0 \quad \text{in } \mathcal{M}; \\
W^\varepsilon &= 1 \quad \text{on } \partial \mathcal{M},
\end{align*}
$$

(7.9)

where $\Delta_p$ denotes the $p$–laplacian and

$$
\beta^\varepsilon = \frac{g_m (\varepsilon - d_\varepsilon)^p}{\alpha_m}.
$$

(7.10)

The functional $b^{\varepsilon,h}(z)$ in our case has the form:

$$
b^{\varepsilon,h}(z) = \inf_{w^\varepsilon} \mu^\varepsilon \int_{K_h^z} \left\{ a^\varepsilon(x) |\nabla w^\varepsilon|^p + g_1^\varepsilon |w^\varepsilon|^p + h^{-p-\gamma} 1_f^\varepsilon |w^\varepsilon - 1|^p \right\} dx,
$$

(7.11)

where the function $a^\varepsilon$ is defined in (7.2) and the infimum is taken over $w^\varepsilon \in W^{1,p}(K_h^z)$. We seek for a function $w^\varepsilon$ minimizing (7.11) in the following form:

$$
w^\varepsilon(x) = \vartheta^\varepsilon(x) + \zeta^\varepsilon(x),
$$

(7.12)

where

$$
\vartheta^\varepsilon(x) = \begin{cases} W^\varepsilon \left( \frac{x - x_0^\varepsilon}{\varepsilon - d_\varepsilon} \right) \quad \text{in } M^\varepsilon_0 \cap K_h^z; \\
1 \quad \text{in } \Omega^\varepsilon_0 \cap K_h^z.
\end{cases}
\quad (7.13)
$$

Then

$$
b^{\varepsilon,h}(z) = \mu^\varepsilon \int_{K_h^z} \left\{ a^\varepsilon(x) |\nabla \vartheta^\varepsilon + \nabla \zeta^\varepsilon|^p + g_1^\varepsilon |\vartheta^\varepsilon + \zeta^\varepsilon|^p + h^{-p-\gamma} 1_f^\varepsilon |\vartheta^\varepsilon + \zeta^\varepsilon - 1|^p \right\} dx. 
$$

(7.14)

We will prove that the function $\zeta^\varepsilon$ gives a vanishing contribution (as $\varepsilon \to 0$ and $h \to 0$) in (7.11). Since the function $w^\varepsilon = \vartheta^\varepsilon + \zeta^\varepsilon$ minimizes the functional (7.11), and $\vartheta^\varepsilon = 1$ in $\Omega^\varepsilon_0$, we have

$$
b^{\varepsilon,h}(z) \leq \mu^\varepsilon \int_{K_h^z} \left\{ a^\varepsilon(x) |\nabla \vartheta^\varepsilon|^p + g_1^\varepsilon |\vartheta^\varepsilon|^p \right\} dx \equiv \Theta^{\varepsilon,h}(z). 
$$

(7.15)

Now let us estimate the functional (7.11) from below. To this end we make use of the following inequality:

$$
|\xi_1 + \xi_2|^p \geq |\xi_1|^p + \delta_p |\xi_2|^p + p|\xi_1|^{p-2}(\xi_1, \xi_2),
$$

(7.16)

where $\xi_1, \xi_2$ are arbitrary vectors from the space $\mathbb{R}^n (n = 2, 3)$, $0 < \delta_p \leq 1$ (here $\delta_p = 1$ when $p = 2$). We have

$$
b^{\varepsilon,h}(z) \geq \Theta^{\varepsilon,h}(z) + \delta_p \mu^\varepsilon \int_{K_h^z} \left\{ a^\varepsilon(x) |\nabla \zeta^\varepsilon|^p + g_1^\varepsilon |\zeta^\varepsilon|^p + h^{-p-\gamma} 1_f^\varepsilon |\zeta^\varepsilon|^m \right\} dx +
$$
\[ +p \mu^\varepsilon \int_{K_h^\varepsilon} \left\{ a^\varepsilon(x) |\nabla \vartheta^\varepsilon|^p - 2(\nabla \vartheta^\varepsilon, \nabla \zeta^\varepsilon) + g 1^\varepsilon_m \vartheta^\varepsilon |\nabla \vartheta^\varepsilon|^p - 2 \zeta^\varepsilon \right\} \, dx. \]  

(7.17)

Now it follows from (7.9), (7.15) and (7.17) that

\[ \Upsilon^{\varepsilon,h}[\zeta^\varepsilon] \equiv \mu^\varepsilon \int_{K_h^\varepsilon} \left\{ a^\varepsilon(x) |\nabla \zeta^\varepsilon|^p + g 1^\varepsilon_m |\zeta^\varepsilon|^p + h^{-p-\gamma} 1^\varepsilon_f |\zeta^\varepsilon|^p \right\} \, dx \leq \]

\[ \leq \frac{\alpha_m p \varepsilon^\theta \mu^\varepsilon}{\delta_p} \int_{K_h^\varepsilon \cap \Omega_{\varepsilon}^\varepsilon_m} \frac{\partial \vartheta^\varepsilon}{\partial \nu} \left| \nabla \vartheta^\varepsilon |\vartheta^\varepsilon|^p - 2 \right| \, d\sigma. \]  

(7.18)

It is easy to see that, for any \( v \in W^{1,p}(\Omega) \), the following inequality holds:

\[ \int_{K_h^\varepsilon \cap \Omega_{\varepsilon}^\varepsilon_m} |v|^p \, d\sigma \leq C \left( \frac{1}{d^\varepsilon} \int_{K_h^\varepsilon \cap \Omega_{\varepsilon}^\varepsilon_f} |v|^p \, dx + d_{\varepsilon}^{p-1} \int_{K_h^\varepsilon \cap \Omega_{\varepsilon}^\varepsilon_f} |\nabla v|^p \, dx \right), \]  

(7.19)

where \( C \) is a constant independent of \( \varepsilon \). Then from (7.18), (7.19) and Hölder’s inequality we obtain

\[ \Upsilon^{\varepsilon,h}[\zeta^\varepsilon] \leq C \mu^\varepsilon \varepsilon^\theta \left( \sum_i \int_{K_h^\varepsilon \cap \partial M_i^\varepsilon} \left\{ \left| \frac{\partial \vartheta^\varepsilon}{\partial \nu} \right| \left| \nabla \vartheta^\varepsilon |\vartheta^\varepsilon|^p - 2 \right| \right\} \, d\sigma \right)^{\frac{p^\varepsilon-1}{p^\varepsilon}} \times \]

\[ \times \left( \frac{1}{d^\varepsilon} \int_{K_h^\varepsilon \cap \Omega_{\varepsilon}^\varepsilon_f} |\zeta^\varepsilon|^p \, dx + d_{\varepsilon}^{p-1} \int_{K_h^\varepsilon \cap \Omega_{\varepsilon}^\varepsilon_f} |\nabla \zeta^\varepsilon|^p \, dx \right)^{\frac{1}{p^\varepsilon}}. \]  

(7.20)

The estimate of the first factor in the right-hand side relies on the following lemma.

**Lemma 7.2** Let \( \vartheta^\varepsilon \) be defined by (7.13), where \( W^\varepsilon \) is the solution of problem (7.9). Then we have

\[ |\nabla \vartheta^\varepsilon| + \left| \frac{\partial \vartheta^\varepsilon}{\partial \nu} \right| \leq C \varepsilon^{-\theta/p} \text{ on } \partial M_i^\varepsilon. \]  

(7.21)

**Proof of Lemma 7.2.** For the sake of notation simplicity, we assume that \( M_i^\varepsilon = \{ x \in \mathbb{R}^2 : 0 < x_k < (\varepsilon - d_\varepsilon) \} \). It follows from (7.9) that \( \vartheta^\varepsilon \) satisfies

\[ \begin{cases} \Delta \vartheta^\varepsilon - \tilde{\beta}^\varepsilon |\vartheta^\varepsilon|^p - 2 \vartheta^\varepsilon = 0 & \text{in } M_i^\varepsilon; \\ \vartheta^\varepsilon = 1 & \text{on } \partial M_i^\varepsilon, \end{cases} \]  

(7.22)

where \( \tilde{\beta}^\varepsilon = \frac{a}{\alpha_m \varepsilon}. \)

Consider the function

\[ v^\varepsilon = \exp \left\{ - \left( \frac{\tilde{\beta}^\varepsilon}{p-1} \right)^{1/p} x_1 \right\}. \]  

(7.23)

It is clear that \( v^\varepsilon(x) \) satisfies the equation in (7.22) and \( v^\varepsilon(x) = 1 \) on the face \( \{ x_1 = 0 \} \).
Then, according to the comparison principle for quasilinear elliptic equations (see, e.g., [15]), the function \((\varphi^\varepsilon - v^\varepsilon)(x)\) attains its positive maximum (or negative minimum) on \(\partial M^\varepsilon_i\). Since the function \((\varphi^\varepsilon - v^\varepsilon)(x)\) equals zero on the face \(\{x_1 = 0\}\) and it is positive on the other faces of the cube \(M^\varepsilon_i\), we have

\[
\varphi^\varepsilon - v^\varepsilon \geq 0, \quad x \in M^\varepsilon_i. \tag{7.24}
\]

On the other hand, \(\varphi^\varepsilon \leq 1\) in \(M^\varepsilon_i\). Then it follows from (7.24) that

\[
1 - v^\varepsilon \geq 1 - \varphi^\varepsilon \geq 0, \quad \text{in } M^\varepsilon_i, \tag{7.25}
\]

and since \(v^\varepsilon(0, x_2) = \varphi^\varepsilon(0, x_2) = 1\), we get

\[
\frac{v^\varepsilon(0, x_2) - v^\varepsilon(\delta, x_2)}{\delta} \geq \frac{\varphi^\varepsilon(0, x_2) - \varphi^\varepsilon(\delta, x_2)}{\delta} \geq 0, \quad \delta > 0. \tag{7.26}
\]

Passing to the limit in (7.26) as \(\delta \to 0\) we obtain

\[
0 \leq \frac{\partial \varphi^\varepsilon}{\partial \nu} \leq \frac{\partial v^\varepsilon}{\partial \nu} \quad \text{on} \quad \{x_1 = 0\},
\]

with

\[
\frac{\partial v^\varepsilon}{\partial \nu}\bigg|_{x_1=0} = \frac{g}{\alpha_m(p - 1)} \varepsilon^{-\theta/p}.
\]

Moreover, since \(\frac{\partial \varphi^\varepsilon}{\partial x_2} = 0\) on \(\{x_1 = 0\}\), we have that

\[
|\nabla \varphi^\varepsilon|\bigg|_{x_1=0} \leq \frac{g}{\alpha_m(p - 1)} \varepsilon^{-\theta/p}.
\]

Clearly the other faces of \(M^\varepsilon_i\) can be treated in the same way. Lemma 7.2 is proved. \(\square\)

Now it follows from (7.21) that

\[
\sum_i \int_{K^\varepsilon_i \cap \partial M^\varepsilon_i} \left( \frac{\partial \varphi^\varepsilon}{\partial \nu} \right) \left| \nabla \varphi^\varepsilon \right|^{p-2} \frac{\partial \varphi^\varepsilon}{\partial \nu} d\sigma \leq C \cdot \frac{h^2}{\varepsilon^2} \cdot \varepsilon \cdot \left( (\varepsilon^{-\theta/p})^{(p-1)/p} \right) \frac{h^2}{\varepsilon^{\theta+1}} = C \cdot h^2 \frac{\varepsilon^{\theta+1}}{\varepsilon}. \tag{7.27}
\]

Then it is easy to show that

\[
\Upsilon^\varepsilon_h[\zeta^\varepsilon] \leq Ch^{3+\frac{2}{p}} \cdot \left( \Upsilon^\varepsilon_h[\zeta^\varepsilon] \right)^{1/p}
\]

for \(\varepsilon\) sufficiently small. Therefore, the function \(\zeta^\varepsilon\) gives a vanishing contribution in the functional \(b^{\varepsilon,h}(z)\), namely

\[
\lim_{\varepsilon \to 0} \Upsilon^\varepsilon_h[\zeta^\varepsilon] = o(h^2) \tag{7.28}
\]

as \(h \to 0\). This yields

\[
b^{\varepsilon,h}(z) = \mu^\varepsilon \int_{K^\varepsilon_i} \left\{ a^\varepsilon(x) |\nabla \varphi^\varepsilon|^p + g^\varepsilon \cdot \varphi^\varepsilon |\varphi^\varepsilon|^p \right\} dx + o(h^2) \tag{7.29}
\]
as \( h \to 0 \) for sufficiently small \( \varepsilon \). Thus, by (7.13) and (7.29) we obtain

\[
\lim_{h \to 0} h^{-2} \lim_{\varepsilon \to 0} b^{\varepsilon,h}(z) = \lim_{h \to 0} h^{-2} \lim_{\varepsilon \to 0} b^{\varepsilon,h}(z) = \lim_{\varepsilon \to 0} \mu^\varepsilon g \int_{\mathcal{M}} W^\varepsilon(x)|W^\varepsilon(x)|^{p-2} \, dx, \tag{7.30}
\]

provided that the last limit exists.

Now it remains to obtain an asymptotic formula for the integral in (7.30). Let

\[
U^\varepsilon(x) = \sum_{j=1}^{2} \left\{ V^+_{\varepsilon j}(x) + V^-_{\varepsilon j}(x) \right\} \tag{7.31}
\]

with

\[
V^\pm_{\varepsilon j}(x) = \exp \left\{ \pm \left( \frac{\beta^\varepsilon}{p-1} \right)^{1/p} (x_j \mp 1/2) \right\}, \quad j = 1, 2 \tag{7.32}
\]

and \( \beta^\varepsilon \) defined in (7.10). Following the arguments from Lemma 7.2 in [23] one can show that

\[
\lim_{\varepsilon \to 0} \mu^\varepsilon g \int_{\mathcal{M}} W^\varepsilon(x)|W^\varepsilon(x)|^{p-2} \, dx = \lim_{\varepsilon \to 0} \mu^\varepsilon g \int_{\mathcal{M}} (U^\varepsilon(x))^{p-1} \, dx = \lim_{\varepsilon \to 0} 4\mu^\varepsilon g \int_{\mathcal{M}} (V^+_{1\varepsilon}(x))^{p-1} \, dx. \tag{7.33}
\]

After straightforward computation we have

\[
4\mu^\varepsilon g \int_{\mathcal{M}} (V^+_{1\varepsilon}(x))^{p-1} \, dx = 4 \frac{\varepsilon}{2d\varepsilon^{\theta/p}} \frac{\varepsilon^{\theta/p}}{\varepsilon} (\alpha_m)^{1/p} \left( \frac{g}{p-1} \right)^{p-1} + o(1) \quad (\varepsilon \to 0).
\]

Thus

\[
b(z) = b = \lim_{\varepsilon \to 0} \mu^\varepsilon g \int_{\mathcal{M}} W^\varepsilon(x)|W^\varepsilon(x)|^{p-2} \, dx = \frac{2(\alpha_m)^{1/p}}{d} \left( \frac{g}{p-1} \right)^{p-1}. \tag{7.34}
\]

Moreover, since the solution \( u \) of (7.4) is a smooth function in \( \Omega \), it follows from Remark 1 that

\[
\lim_{\varepsilon \to 0} \frac{1}{\text{meas } \Omega^\varepsilon_f} \| u^\varepsilon - u \|_{W^1_p(\Omega)}^p = 0.
\]

Theorem 7.1 is proved.

\[\square\]

**Remark 3** Let us notice that if the thickness \( d \varepsilon \gg \varepsilon^{\theta/p} \) then \( \{ u^\varepsilon \} \), the sequence of solutions of problem (7.1), \( D^p_{\Omega_f} \)-converges to \( u \) the solution of the following problem:

\[
\int_{\Omega} \left\{ \frac{\alpha_f}{2}(|u_{x_1}|^p + |u_{x_2}|^p) + g|u|^p - pS(x)u \right\} \, dx \to \inf \ u \in W^{1,p}(\Omega). \tag{7.35}
\]

This case corresponds to a model where the process is governed only by the fissures system.
7.2 3D periodic example

Let $\Omega = \Omega_f^\varepsilon \cup \overline{\Omega_m^\varepsilon}$ be a bounded domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial \Omega$. Following [13], we assume that the fissures system $\Omega_f^\varepsilon$, i.e. the highly permeable material is distributed in thin orthogonal layers of thickness $d_\varepsilon = d\varepsilon^{\theta/p}$, $(d > 0, \theta > p \geq 2)$ and the matrix part $\Omega_m^\varepsilon$ is made of low permeable cubic porous blocks $\mathcal{M}_i^\varepsilon$ centered at $x_i^\varepsilon \in \Omega$. The centers $x_i^\varepsilon$ are periodically, with the period $\varepsilon$, distributed in $\Omega$ (see Figure 1).

![Figure 1. A 3D example of the microstructure of the domain $\Omega$.](image)

Consider the variational problem (7.1)--(7.2). The homogenization result for this example is the following.

**Theorem 7.3** Let $\{u^\varepsilon\}$ be the sequence of solutions of problem (7.1)--(7.2). Then $\{u^\varepsilon\}$ $D_{\Omega_f^\varepsilon}^p$–converges to the solution $u$ of the problem:

$$
\int_{\Omega} \{A(\nabla u) + B|u|^p - pS(x)u\} \, dx \rightarrow \inf \quad u \in W^{1,p}(\Omega),
$$

where

$$
A(\nabla u) = \frac{\alpha f}{3} \left( \left( |u_{x_1}|^2 + (u_{x_2})^2 \right)^{\frac{p}{2}} + \left( |u_{x_1}|^2 + (u_{x_3})^2 \right)^{\frac{p}{2}} + \left( |u_{x_2}|^2 + (u_{x_3})^2 \right)^{\frac{p}{2}} \right)
$$

and $B$ is given by (7.5).

The proof of Theorem 7.3 is similar to that of Theorem 7.1.
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References


