Convergence analysis of a new BEM–FEM coupling method for two dimensional fluid–solid interaction problems

Salim Meddahi*

Universidad de Oviedo, Departamento de Matemáticas, Oviedo, Spain

Abstract

We present a new numerical method, based on a coupling of finite elements and boundary elements, to solve a fluid–solid interaction problem posed in the plane. The boundary unknowns involved in our formulation are approximated by a spectral method.

We provide error estimates for the Galerkin method, propose fully discrete schemes based on elementary quadrature formulas and show that the perturbation due to this numerical integration preserves the optimal rate of convergence. We also suggest an iterative method to solve the complicated linear systems of equations that arise from this type of approximation schemes.

Keywords: exterior boundary value problems, Helmholtz equation, elastodynamic equation, integral equations, finite elements, spectral methods

1 Introduction

We introduce in this paper a new numerical scheme to compute the scattered waves and the elastic vibrations that take place in the interaction between a bounded solid body and the compressible inviscid fluid that surround it.

The numerical difficulties related to the fact that the scattered wave propagates in an unbounded region is overcome by imposing absorbing boundary conditions on an artificial boundary containing the obstacle. This permits one to incorporate the far-field effects into a finite element discretization of the problem in a bounded region. The absorbing boundary conditions may be of local (differential) or global type; we refer to [8, 6] for a review of such methods.

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In this paper, we follow [3, 7] and use linear integral equations as nonlocal boundary conditions on the artificial interface. We signalize that in [3, 7] the interface that separates the two mediums (the wet interface) is used as a coupling boundary. In this case, the well posedness of the resulting formulation (at the continuous level) requires regularity assumptions that may not be fulfilled in practice by the wet interface. Here, we impose the absorbing boundary conditions on a smooth but arbitrary interface that contains the obstacle in its interior. This enlarges a little the domain of finite element computations but this drawback is compensated by the fact that we remove the limitation to problems with smooth wet boundaries.

The presence of integrals with nearly singular integrands augurs that the matrix assembly process is a delicate operation in all the BEM-FEM coupling procedures. The design of efficient algorithms for this task is of great importance in order to improve the practicability of these methods. Here, we follow the recent technique introduced in [12, 14, 13, 15] and change all terms on the interface to periodic functions by means of a smooth parameterization of the artificial boundary. This simple idea allows one to approximate the weakly singular boundary integrals by elementary quadrature formulas. Furthermore, as shown in [13], they permit one to approximate the periodic representation of the unknowns defined on the boundary by trigonometric polynomials.

However, the method presented in [13] is limited to a finite element method of first order. Moreover, the convergence of the fully discrete scheme is only quasi-optimal since the leading term in the asymptotic behavior of the error is found to be $O(\sqrt{\log n} h)$ (see, Theorem 7.5 in [13]) where $h$ and $n$ are the finite element and the spectral discretization parameters. In this paper, we introduce a BEM-FEM method that permits one to use an arbitrary finite element order $m$ and the corresponding fully discrete scheme maintain the optimal order of convergence, i.e. $O(h^m)$.

The advantage of a hybrid scheme that combines a finite element method with a spectral method is that one may reduce drastically the degrees of freedom on the interface without affecting the convergence of the method. This fact is confirmed by the numerical experiments performed in [13, 15]. It is then possible to eliminate the periodic unknown at matricial level by a static condensation process and reduce by the way the complexity of the linear systems. We suggest here a preconditioned GMRES method to solve the reduced linear system of equations. The resulting iterative method only requires the solution of standard (interior) elliptic finite element problems. It also allows one to avoid storing the huge global matrix.

The paper is organized as follows. In the first part, which consists of sections 2, 3, 4 and 5, we introduce the model problem, derive its BEM-FEM formulation and prove its well-posedness. In the second part of the paper (sections 6 and 7) we introduce the Galerkin discretization of the BEM-FEM method and give its convergence analysis. Finally, in sections 8 and 9, we describe the quadrature rules that we use to obtain the fully discrete schemes and analyze the resulting completely discrete problems.
1.1 Notations and Sobolev spaces

In the sequel, we deal with complex valued functions and the symbol \( i \) is used for \( \sqrt{-1} \). We denote by \( \overline{\alpha} \) the conjugate of a complex number \( \alpha \in \mathbb{C} \) and by \( |\alpha| \) its modulus. Let \( \Omega \) be a bounded open set of \( \mathbb{R}^2 \). We denote by \( \| \cdot \|_{0,\Omega} \) the \( L^2(\Omega) \)-norm corresponding to the inner product \( \int_\Omega f \overline{g} \, dx \). More generally, for any \( m \in \mathbb{N} \), \( \| \cdot \|_{m,\Omega} \) stands for the norm of the Sobolev space \( H^m(\Omega) \), see [4].

On the other hand, we will also consider periodic Sobolev spaces. Let \( C^\infty_{2\pi} \) be the space of \( 2\pi \)-periodic and infinitely differentiable complex valued functions of a single variable. Given \( g \in C^\infty_{2\pi} \), we define its Fourier coefficients
\[
\hat{g}(k) := \frac{1}{2\pi} \int_0^{2\pi} g(s)e^{-iks} \, ds.
\]
Then, for \( p \in \mathbb{R} \), we define the Sobolev space \( H^p \) to be the completion of \( C^\infty_{2\pi} \) with the norm
\[
\|g\|_p := \left( \sum_{k \in \mathbb{Z}} (1 + |k|^2)^p |\hat{g}(k)|^2 \right)^{1/2}.
\]
It is well known (see [9]) that \( H^p \) are Hilbert spaces and \( H^p \subset H^q \) for every \( p > q \), the inclusion being dense and compact. Moreover, the \( H^0 \)-bilinear \( \int_0^{2\pi} \lambda(t) \eta(t) \, dt \) can be extended to represent the duality between \( H^{-p} \) and \( H^p \) for all \( p > 0 \). We will keep the same notation for this duality bracket.

Throughout this paper \( C \) will denote positive constants, not necessarily the same at different occurrences, which are independent of the parameters \( h \) and \( n \) and functions involved.

2 Physical assumptions and governing equations

We are concerned with the interaction between an elastic body and a fluid that fills the space around it. We suppose that a wave is incident upon the body and we are required to determine its response and the scattered wave.

We assume that the obstacle is an infinitely long cylinder parallel to the \( x_3 \)-axis whose cross section is \( \Omega_s \). We denote by \( \Sigma \) the boundary of \( \Omega_s \). The incident acoustic wave and the volume force acting on the obstacle are supposed to exhibit a time–harmonic behavior with frequency \( \omega \). We will denote their amplitudes \( w = w(x_1, x_2) \) and \( f = f(x_1, x_2) \), respectively. The incident wave is generally taken to satisfy the Helmholtz equation
\[
\Delta w + k^2 w = 0 \quad \text{in} \quad \Omega_f := \mathbb{R}^2 \setminus \overline{\Omega_s}.
\]

The phenomenon is invariant under a translation in the \( x_3 \)-direction. Then, we may consider a bidimensional model posed in the frequency domain. The unknowns of the problem are the amplitude \( u : \Omega_s \to \mathbb{C}^2 \) of the solid displacements field and the amplitude \( p : \Omega_f \to \mathbb{C} \) of the scattered pressure.
We suppose that the solid is isotropic and linearly elastic, with mass density \( \rho_s \) and Lamé moduli \( \lambda, \mu \). We denote as usual the stress tensor by \( \sigma(u) := \lambda \text{tr} \varepsilon(u) I + 2 \mu \varepsilon(u) \), where \( \varepsilon_{ij}(u) := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \) is the infinitesimal strain tensor. Furthermore, we assume that the fluid is ideal, compressible and homogeneous with mass density \( \rho_f \) and wave number \( k = \frac{\omega}{c} \) where \( c \) is the speed of sound in the linearized fluid.

Let us denote by \( n \) the unit normal on \( \Sigma \) directed into \( \Omega_f \). Under the hypothesis of small oscillations both in the solid and the fluid, \( u \) and \( p \) are found out to satisfy the equations

\[
\begin{align*}
\nabla \cdot \sigma(u) + \rho_s \omega^2 u &= -f \quad \text{in } \Omega_s, \\
\Delta p + k^2 p &= 0 \quad \text{in } \Omega_f, \\
\sigma(u)n &= -(p + w)n \quad \text{on } \Sigma, \\
\rho_f \omega^2 u \cdot n &= \frac{\partial (p + w)}{\partial n} \quad \text{on } \Sigma,
\end{align*}
\]

and the decay condition

\[
\frac{\partial p}{\partial r} - ikp = o(r^{-1/2})
\]

when \( r \to +\infty \) uniformly for all directions \( \frac{\mathbf{e}}{|\mathbf{e}|} \).

We notice that the first two equations of (1) are the acoustic and elastodynamic equations, respectively. The transmission conditions posed on \( \Sigma \) represent the equilibrium of forces (dynamic boundary condition) and the equality of the normal displacements of solid and fluid (kinematic boundary condition). Finally, equation (2) means that the far field absorbs the outgoing waves (cf. [8] for more details).

It is known that if \( f = 0 \) and \( w = 0 \) then \( p = 0 \) and \( u \) is solution of (see [10])

\[
\begin{align*}
\nabla \cdot \sigma(u) + \rho_s \omega^2 u &= 0 \quad \text{in } \Omega_s, \\
\sigma(u)n &= 0 \quad \text{on } \Sigma, \\
u \cdot n &= 0 \quad \text{on } \Sigma.
\end{align*}
\]

It turns out that for certain regions and some frequencies \( \rho_s \omega^2 \), known as Jones frequencies, problem (3) have nontrivial solutions. This seems to be a rare eventuality but we will, in any case, assume that (3) admits only one solution, the trivial one.

### 3 A variational formulation with a nonlocal boundary conditions

Let us introduce an artificial boundary \( \Gamma \) such that \( \Omega_s \) lays in its interior. Then, \( \Gamma \) separates \( \mathbb{R}^2 \) into a bounded domain \( \Omega^- \) and an unbounded region \( \Omega^+_f \) exterior to \( \Gamma \). We denote \( \Omega_f^- := \Omega_f \cap \Omega^- \). Notice that \( \overline{\Omega^+} = \overline{\Omega^+_s} \cup \overline{\Omega^-_f} \); cf. Figure 1.
We consider the bilinear forms

\[ E^{\omega}(u, v) := \int_{\Omega_s} (\sigma(u) : \varepsilon(v) - \rho_s \omega^2 u \cdot v) \, dx, \]
\[ a^k(p, q) := \frac{1}{\rho_f \omega^2} \int_{\Omega_f} (\nabla p \cdot \nabla q - k^2 p q) \, dx \quad \text{and} \quad D(v, q) := \int_{\Sigma} v \cdot n q \, d\tau. \]

It is straightforward that, in \( \Omega_s \), \( u \) satisfies the variational formulation:

\[
\begin{cases}
\text{find } u \in (H^1(\Omega_s))^2 \text{ such that } \\
E^{\omega}(u, v) + D(v, p) = L(v) \quad \forall v \in (H^1(\Omega_s))^2,
\end{cases}
\]

where

\[ L(v) := \int_{\Omega_s} f \cdot v \, dx - D(v, w) \]

while \( p|_{\Omega_f} \) is a solution of

\[
\begin{cases}
\text{find } p \in H^1(\Omega_f) \text{ such that } \\
a^k(p, q) + \rho_f \omega^2 D(u, q) - \int_{\Gamma} \frac{\partial p}{\partial n} q \, d\tau = \ell(q) \quad \forall q \in H^1(\Omega_f).
\end{cases}
\]

Here, the unit normal \( \nu \) on \( \Gamma \) is directed into \( \Omega_f^+ \) and

\[ \ell(q) := \int_{\Sigma} \frac{\partial w}{\partial n} q \, d\tau. \]

The solution in the exterior domain \( \Omega_f^+ \) is represented in the form

\[ p(x) = \int_{\Gamma} \frac{\partial E(x, y)}{\partial \nu_y} \psi(y) \, d\sigma_y \quad \forall x \in \Omega_e, \]
where

$$E(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|)$$

is the radial outgoing fundamental solution of the Helmholtz equation and $H_0^{(1)}$ stands for the Hankel function of order 0 and first type.

Notice that $p$ satisfies the Helmholtz equation $\Delta p + k^2 p = 0$ in $\Omega_f^+$ and the radiation condition (2) for all density $\psi : \Gamma \to \mathbb{C}$. Thus, if we want to obtain a solution of the global problem (1), we have just to impose the following transmission conditions on $\Gamma$

$$p|\Gamma^- = \left( \int_{\Gamma} \frac{\partial E(x, y)}{\partial \nu_y} \psi(y) \, d\sigma_y \right)|_{\Gamma^-},$$

$$\frac{\partial p}{\partial \nu}|\Gamma^- = \frac{\partial}{\partial \nu}(x \mapsto \int_{\Gamma} \frac{\partial E(x, y)}{\partial \nu_y} \psi(y) \, d\sigma),$$

where $\Gamma^+$ and $\Gamma^-$ denote the exterior and interior sides of $\Gamma$ respectively.

Let us first introduce some notations and basic properties. In what follows, we choose $\Gamma$ to be an infinitely differentiable boundary and we denote by $x : \mathbb{R} \to \mathbb{R}^2$ a regular $2\pi$-periodic parametric representation of this curve:

$$|x'(s)| > 0 \quad \forall s \in \mathbb{R} \quad \text{and} \quad x(s) = x(t) \quad \text{iff} \quad t - s \in 2\pi\mathbb{Z}.$$

Therefore, we can identify any function defined on $\Gamma$ with a $2\pi$-periodic function. We can also define the parameterized trace on $\Gamma$ as the linear continuous extension of

$$\gamma : C^\infty(\overline{\Omega}) \to L^2(0, 2\pi)$$

$$u \mapsto \gamma u(\cdot) := u|\Gamma(x(\cdot))$$

to $H^1(\Omega)$. The resulting linear application $\gamma : H^1(\Omega) \to H^{1/2}$ is bounded and onto; cf. Theorem 8.15 of [9].

We introduce the parameterized versions of the simple and double layer acoustic potentials

$$Sg(s) := \int_0^{2\pi} V(s, t)g(t)dt \quad \text{and} \quad Dg(s) := \int_0^{2\pi} K(s, t)g(t)dt,$$

where

$$V(s, t) := \frac{i}{4} H_0^{(1)}(k|x(s) - x(t)|)$$

and

$$K(s, t) := -\frac{k^2}{4} H_1^{(1)}(k|x(t) - x(s)|) \frac{x'(t)(x_1(t) - x_1(s)) - x_1'(t)(x_2(t) - x_2(s))}{|x(t) - x(s)|}$$

with $H_1^{(1)}$ being the Hankel function of first type and order one.
We also introduce the hypersingular operator $H$ which is related to the single layer operator via tangential derivatives, see [11]. With our notations this relation reads:

$$\int_0^{2\pi} \eta(H\psi)\,dt = \int_0^{2\pi} \eta'(S\psi')\,dt - k^2 \int_0^{2\pi} \eta(\tilde{S}\psi)\,dt \quad \forall \psi, \eta \in H^{1/2},$$

(8)

here $\tilde{S}$ is the integral operator whose kernel is given by

$$\tilde{V}(t,s) := x'(t) \cdot x'(s) V(t,s).$$

Using the classical jump conditions for the double layer potential and parameterizing the transmission conditions (7), yields:

$$\gamma_p = \left( \frac{1}{2} I + D \right) \psi,$$

(9)

$$\xi = -H\psi,$$

(10)

where $I$ is the identity operator in $H^{1/2}$ and the auxiliary unknown $\xi$ is given in terms of the normal derivative of $p$ on $\Gamma$ by

$$\xi := |x'| \frac{\partial p}{\partial \nu} \circ x.$$

Combining (4) and (5) with a variational versions of (9) and (10) we arrive at the following global weak formulation of (1)–(2):

$$\begin{aligned}
\begin{cases}
\text{find } u \in (H^1(\Omega_s))^2, \ p \in H^1(\Omega_f), \ \xi \in H^{-1/2} \ \text{and } \psi \in H^{1/2} \ \text{such that}
\end{cases}

E^\omega(u, v) + D(v, p) = L(v) \quad \forall v \in (H^1(\Omega_s))^2 \\
a_k(p, q) + \rho f \omega^2 D(u, q) - \int_0^{2\pi} \xi \gamma q\,dt = \ell(q) \quad \forall q \in H^1(\Omega_f^-) \\
\int_0^{2\pi} \mu \gamma p\,dt - \frac{1}{2} \int_0^{2\pi} \mu \psi\,dt - b(\psi, \mu) = 0 \quad \forall \mu \in H^{-1/2} \\
\int_0^{2\pi} \xi \varphi\,dt + c(\psi, \varphi) = 0 \quad \forall \varphi \in H^{1/2}
\end{aligned}$$

(11)

where

$$b(v, \mu) := \int_0^{2\pi} \mu(t)(D\gamma v)(t)\,dt \quad \text{and} \quad c(\xi, \mu) := \int_0^{2\pi} \mu(t)(H\xi)(t)\,dt.$$

We introduce the Hilbert space

$$\mathbf{M} = H^1(\Omega_s) \times H^1(\Omega_f^-) \times H^{-1/2} \times H^{1/2}$$

and denote by $\mathbf{M}'$ its dual space pivotal to $L^2(\Omega_s) \times L^2(\Omega_f^-) \times L^2(0, 2\pi) \times L^2(0, 2\pi)$. In what follows, the bracket $[\cdot, \cdot]$ represents the duality between $\mathbf{M}$ and $\mathbf{M}'$. 
We also introduce the operators $E^{\omega} : H^1(\Omega_s)^2 \rightarrow (H^1(\Omega_s)^2)', A_k : H^1(\Omega_f^-) \rightarrow (H^1(\Omega_f^-))'$ and $B : H^1(\Omega_f^-) \rightarrow (H^1(\Omega_s)^2)'$ by
\[
\langle E^{\omega}u, v \rangle_{H^1(\Omega_s) \times H^1(\Omega_s)} = E^{\omega}(u,v) \quad \forall u, v \in H^1(\Omega_s)^2
\]
\[
\langle A_k p, q \rangle_{H^1(\Omega_f^-) \times H^1(\Omega_f^-)} = a_k(p,q) \quad \forall p, q \in H^1(\Omega_f^-)
\]
and
\[
\langle B q, v \rangle_{H^1(\Omega_s) \times H^1(\Omega_s)} = D(v,q) \quad \forall q \in H^1(\Omega_f^-), \; v \in H^1(\Omega_s)^2.
\]
Notice that
\[
A = \begin{pmatrix}
E^{\omega} & B & 0 & 0 \\
\rho f^{\omega} 2B^* & A_k & -\gamma^* & 0 \\
0 & \gamma & 0 & -(\frac{1}{2}I + D) \\
0 & 0 & I^* & \mathcal{H}
\end{pmatrix} : M \rightarrow M'
\]
is the operator corresponding to the bilinear form of problem (11), where the superscript $(\cdot)^*$ denotes transposition of operators. For example, $\gamma^* : H^{-1/2} \rightarrow (H^1(\Omega))'$ is defined in terms of $\gamma$ by
\[
\langle \gamma^* \mu, q \rangle_{H^1(\Omega_f^-) \times H^1(\Omega_f^-)} = \int_0^{2\pi} \mu \gamma q \, dt \quad \forall q \in H^1(\Omega_f^-) \quad \forall \mu \in H^{-1/2}.
\]

4 Properties of the integral operators

We will give a brief account of some fundamental tools which concern the properties of $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{H}$ when mapping between Sobolev spaces. Let us first introduce the parameterized versions of the simple layer harmonic potential, that is
\[
\mathcal{S}_0 g(t) = \int_0^{2\pi} V_0(t, s) g(s) \, ds
\]
with
\[
V_0(t, s) := -\frac{1}{2\pi} \log |x(t) - x(s)|.
\]
In addition, as shown in [9], the corresponding harmonic hypersingular operator $\mathcal{H}_0$ is related to the simple layer operator via tangential derivatives as follows
\[
\int_0^{2\pi} (\mathcal{H}_0 \psi) \eta \, dt = \int_0^{2\pi} \eta' (\mathcal{S}_0 \psi') \, dt \quad \forall \psi, \eta \in H^{1/2}. \quad (12)
\]
Lemma 1. For any \( p \in \mathbb{R} \) the mappings \( S : H^p \to H^{p+1} \), \( D : H^p \to H^{p+2} \) and \( S - S_0 : H^p \to H^{p+3} \) are bounded. Furthermore, \( S_0 \) is \( H_0^{-1/2} \)-elliptic, i.e., there exists \( \alpha_0 > 0 \) such that
\[
\int_0^{2\pi} \eta(S_0 \eta) \, dt \geq \alpha_0 \| \eta \|_{-1/2}^2 \quad \forall \, \eta \in H_0^{-1/2},
\]
where \( H_0^{-1/2} := \{ \eta \in H^{-1/2} : \int_0^{2\pi} \eta \, dt = 0 \} \).

Proof. For \( n = 0, 1, 2 \) we introduce the auxiliary integral operators
\[
\Lambda_n(\xi)(t) := -\frac{1}{2\pi} \int_0^{2\pi} \left( \sin \frac{t-s}{2} \right)^n \log \left( \frac{4}{e} \sin \frac{t-s}{2} \right) \xi(s) \, ds.
\]
Let \( f_m(t) := \exp(\im t m) \). One may deduce easily from the property (see [9])
\[
\Lambda_0 f_m = \frac{1}{\max(1,|m|)} f_m \quad (m \in \mathbb{Z})
\]
that \( \Lambda_0 \) is a pseudodifferential operator of order -1, i.e., \( \Lambda_0 : H^p \to H^{p+1} \) is bounded for all \( p \in \mathbb{R} \).

In general, given a function \( D(t,s) \) which is in \( C_2^{\infty} \) with respect to each variable, it is shown in [16] that \( \mu \mapsto \Lambda_n(D(t,\cdot)\mu(\cdot)) \) is a pseudodifferential operator of order \(-n-1\). Therefore, the results for \( S \), \( D \) and \( S - S_0 \) are obtained by noticing that there exist two functions \( D_1 \) and \( D_2 \) that belong to \( C_2^{\infty} \) in each of their two variables such that
\[
(S_0 \mu)(t) = (\Lambda_0 \mu)(t) + (\mathcal{F}_0 \mu)(t) \quad (S \mu)(t) = (\Lambda_0 \mu)(t) + \Lambda_2(D_1(t,\cdot)\mu(\cdot)) + (\mathcal{F} \mu)(t)
\]
and
\[
(D \mu)(t) = \Lambda_1(D_2(t,\cdot)\mu(\cdot)) + (\mathcal{G} \mu)(t)
\]
where \( \mathcal{F}_0 \), \( \mathcal{F} \) and \( \mathcal{G} \) are integral operators with \( 2\pi \)-periodic and infinitely differentiable kernels.

On the other hand, the ellipticity of \( S_0 \) on \( H_0^{-1/2} \) is an easy consequence of the corresponding result in the real case (see, e.g. [9]) and the fact that the function \( V_0 \) is real and symmetric.

Finally, as a consequence of Lemma 1, identity (12), and the fact that the derivative (acting from \( H_0^{1/2} \) into \( H_0^{-1/2} \)) is an isomorphism, we deduce that \( \mathcal{H}_0 \) is elliptic on \( H_0^{1/2} \), that is there exists \( \alpha > 0 \) such that
\[
\int_0^{2\pi} (\mathcal{H}_0 \eta) \overline{\eta} \, dt \geq \alpha \| \eta \|_{1/2}^2 \quad \forall \, \eta \in H_0^{1/2}.
\]
5 Existence and uniqueness

We introduce the auxiliary operator

\[ A_0 = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & A & -\gamma^* & 0 \\ 0 & \gamma & 0 & -\frac{1}{2}I \\ 0 & 0 & I^* & H \end{pmatrix} \]  \hspace{1cm} (16) \]

where

\[ \langle Eu, v \rangle_{H^1(\Omega_s) \times H^1(\Omega_s)} := \int_{\Omega_s} (\sigma(u) : \varepsilon(v) + u \cdot v) \, dx, \]

and

\[ \langle Ap, q \rangle_{H^1(\Gamma) \times H^1(\Gamma)} = \int_{\Gamma} \nabla p \cdot \nabla q \, dx + \left( \int_0^{2\pi} \gamma p \, dt \right) \left( \int_0^{2\pi} \gamma q \, dt \right). \]

**Proposition 2.** If there exists a constant \( C > 0 \) such that for all \( \hat{L} \in M' \) and for all \( \hat{u} \in M \) satisfying \( A_0 \hat{u} = \hat{L} \) we have

\[ \| \hat{u} \|_M \leq C' \| \hat{L} \|_{M'}, \]

then \( A_0 : M \to M' \) is an isomorphism.

**Proof.** We deduce immediately from the hypothesis that

\[ \| \hat{u} \|_M \leq C \| A_0 \hat{u} \|_{M'} \hspace{1cm} \forall \hat{u} \in M. \]

Thus, \( A_0 \) satisfies the Inf-Sup condition

\[ \sup_{\hat{v} \in M} \frac{|[A_0 \hat{u}, \hat{v}]_{M' \times M}|}{\| \hat{v} \|_M} \geq \frac{1}{C} \| \hat{u} \|_M \hspace{1cm} \forall \hat{u} \in M. \]  \hspace{1cm} (17) \]

We conclude by noting that we can convert \( A_0 \) into a self-adjoint operator by just multiplying the second line of matrix (16) by 2 and the third one by \(-2.\)

Let us now prove that the hypothesis of the last proposition holds true.

**Lemma 3.** There exists a constant \( C > 0 \) such that for all \( \hat{L} \in M' \) and for all \( \hat{u} \in M \) satisfying \( A_0 \hat{u} = \hat{L} \) we have

\[ \| \hat{u} \|_M \leq C' \| \hat{L} \|_{M'}. \]
Proof. We assume here that the components of $\hat{L}$ are given by $(L, \ell, f, g) \in M'$ and let $\hat{u} = (u, p, \xi, \psi) \in M$ be a solution of

$$A_0 \hat{u} = \hat{L}.$$ 

Testing the last equation with $\hat{v} = (0, 1, -2, 2) \in M$ yields

$$2(\pi - 1) \int_0^{2\pi} \gamma p \, dt + \int_0^{2\pi} \xi \, dt + \int_0^{2\pi} \psi \, dt = \ell(1) + 2 \int_0^{2\pi} g \, dt - 2 \int_0^{2\pi} f \, dt.$$ 

Let

$$V = \{q \in H^1(\Omega_\ell^-); \quad \int_0^{2\pi} \gamma q \, dt = 0\}$$

and denote $M_0 := H^1(\Omega_\ell^-)^2 \times V \times H_0^{-1/2} \times H_0^{1/2}$. We introduce the function $\phi = \psi - f$ and define

$$p_0 = p - \frac{1}{2\pi} \int_0^{2\pi} \gamma p \, dt, \quad \xi_0 = \xi - \frac{1}{2\pi} \int_0^{2\pi} \xi \, dt \quad \text{and} \quad \phi_0 = \phi - \frac{1}{2\pi} \int_0^{2\pi} \phi \, dt.$$ 

It is straightforward that $\hat{u}_0 := (u, p_0, \xi_0, \phi_0) \in M_0$ satisfies

$$[A_0 \hat{u}_0, \hat{v}] = [\tilde{L}, \hat{v}] \quad \forall \hat{v} \in M_0,$$

where $\tilde{L} = (L, \ell, 0, \tilde{g})$ with $\tilde{g} = g - \mathcal{H}_0 f$.

Now, we choose $\hat{v} = (\bar{u}, 2\overline{p_0}, 2\overline{\xi_0}, \overline{\phi_0})$ in (19) and take the real part of the resulting equation to obtain

$$E(\bar{u}, \bar{u}) + 2|p_0|^2_{1, \Omega_\ell^-} + \int_0^{2\pi} (\mathcal{H}_0 \phi_0) \overline{\phi_0} \, dt = \text{Re}[L(\bar{u}) + 2\ell(\overline{p_0}) + \int_0^{2\pi} \tilde{g} \overline{\phi_0} \, dt].$$

Finally, (15), Korn’s inequality, together with the fact that the $H^1(\Omega_\ell^-)$ norm and seminorm are equivalent on $V$ give

$$\|u\|_{1, \Omega_\ell^-} + |p_0|_{1, \Omega_\ell^-} + \|\phi_0\|_{1/2} \leq C_1 \|\tilde{L}\|_{M'}$$

(20)

On the other hand

$$\|\xi_0\|_{-1/2} = \sup_{\varphi \in H_0^{1/2}} \frac{\int_0^{2\pi} \xi_0 \varphi \, dt}{\|\varphi\|_{1/2}} = \sup_{\varphi \in H_0^{1/2}} \frac{\int_0^{2\pi} \overline{\tilde{g}} \varphi \, dt - \int_0^{2\pi} (\mathcal{H}_0 \phi_0) \varphi \, dt}{\|\varphi\|_{1/2}} \leq C_2 \|\tilde{L}\|_{M'}$$

(21)

The Lemma is now a direct consequence of identity (18) and estimates (20) and (21). □

Theorem 4. Assume that $-k^2$ is not an eigenvalue of the Laplacian in $\Omega$ with a Neumann boundary condition on $\Gamma$ and that problem (3) admits only the trivial solution. Then, problem (11) is well posed.
Proof. We deduce from Proposition 2 and Lemma 3 that $A_0 : M \to M'$ is an isomorphism. Moreover, Lemma 1 and the compactness of the canonical injections $H^1(\Omega) \hookrightarrow L^2(\Omega)$, $H^{1/2}(\Sigma) \hookrightarrow L^2(\Sigma)$ and $H^{1/2} \hookrightarrow L^2(0, 2\pi)$ imply that $A - A_0 : M \to M'$ is compact. Hence $A$ is a fredholm operator of index zero and the Theorem reduces to prove uniqueness of solution for (11).

Let $(u_0, p_0, \xi_0, \psi_0)$ be a solution of (11) with $f = 0$ and $w = 0$. We define the function

$$\tilde{p}(x) := \begin{cases} p_0(x) & \text{if } x \in \Omega_f^-, \\ z(x) & \text{if } x \in \Omega_f^+ \end{cases}$$

where

$$z(x) := \int_0^{2\pi} \frac{\partial E}{\partial n}(x, x(t)) \psi_0(t) |x'(t)| \, dt.$$ 

It is easy to show that $u_0, p_0, \xi_0$ and $\psi_0$ solve the equations:

$$\left\{ \begin{array}{ll} \nabla \cdot \sigma(u_0) + \rho_s \omega^2 u_0 &= 0 & \text{in } \Omega_s, \\
\Delta p_0 + k^2 p_0 &= 0 & \text{in } \Omega_f^-, \\
\sigma(u_0) n &= -(p_0 + w) n & \text{on } \Sigma, \\
\rho_f \omega^2 u_0 \cdot n &= \frac{\sigma(p_0 + w)}{\sigma_n} & \text{on } \Sigma, \\
\gamma_p &= (\frac{1}{2} I + D) \psi_0, \\
\xi_0 &= -\mathcal{H} \psi_0. \end{array} \right.$$ 

(22)

On the other hand, $z$ also solves the Helmholtz equation in $\Omega_f^+$:

$$\Delta z + k^2 z = 0 \quad \text{in } \Omega_f^+$$ 

(23)

and it satisfies the asymptotic condition (2). Besides, the jump properties of the double layer potential and the normal derivative of the single layer potential through $\Gamma$ provide the relations (cf. [16]):

$$\gamma z = (\frac{1}{2} I + D) \psi_0,$$ 

(24)

$$|x'(t)| \frac{\partial x}{\partial n} \circ x = -\mathcal{H} \psi_0.$$ 

Combining (23), (24) and (2) with (22) proves that $(u_0, \tilde{p})$ is a solution of (1) with data $f = 0$ and $w = 0$. Now, Rellich theorem (cf. [5]) and our assumption on problem (3) ensures that $(u_0, \tilde{p})$ vanishes identically. Consequently, $u_0 = 0, p_0 = 0, \xi_0 = 0$ and

$$\frac{1}{2} I + D) \psi_0 = 0.$$

Theorem 3.3.4. of [16] proves that, under our hypothesis on $k$, operator $\frac{1}{2} I + D$ is one-to-one and the result follows. 

\qed
6 Finite elements with curved triangles

For simplicity of exposition, in the rest of the paper we assume that \( \Sigma \) is a polygonal boundary. Let \( N \) be a given integer. We consider the equidistant subdivision \( \{ t_i := i\pi/N; \ i = 0, \ldots, 2N - 1 \} \) of the interval \([0, 2\pi]\) with \( 2N \) grid points. We denote by \( \Omega_h \) the polygonal domain whose vertices lying on \( \Gamma \) are \( \{ x(t_i): i = 0, \ldots, 2N - 1 \} \). Let \( \{ \tau_h \} \) be a regular family of triangulations of \( \Omega_h \) by triangles \( T \) of diameter \( h_T \) not greater than \( \max |x'(s)|/h \) with \( h := \pi/N \). We assume that the restriction \( \tau^*_h := \{ T \in \tau_h; \ T \subset \Omega_s \} \) of \( \tau_h \) to \( \Omega_s \) is a triangulation and set \( \tau^*_f := \tau_h \setminus \tau^*_h \). Notice that \( \Omega_{f,h} := \text{interior}(\bigcup_{T \in \tau^*_f} T) \) is a polygonal approximation of \( \Omega_f \).

We obtain from \( \tau^*_f \) a triangulation \( \tilde{\tau}^*_f \) of \( \Omega_f \) by replacing each triangle of \( \tau^*_f \) with one side along \( \partial \Omega_h \) by the corresponding curved triangle.

Let \( T \) be a curved triangle of \( \tilde{\tau}^*_f \). We denote its vertices by \( a^1_T, a^2_T \) and \( a^3_T \), numbered in such a way that \( a^2_T \) and \( a^3_T \) are the endpoints of the curved side of \( T \). Let \( t_i, t_{i+1} \in [0, 2\pi] \) be such that \( x(t_i) = a^2_T \) and \( x(t_{i+1}) = a^3_T \). Then, \( \varphi(t) := x(t + t h) \ (t \in [0, 1]) \) is a parameterization of the curved side of \( T \). Let \( \hat{T} \) be the reference triangle with vertices \( \hat{a}_1 := (0, 0), \hat{a}_2 := (1, 0) \) and \( \hat{a}_3 := (0, 1) \). Consider the affine map \( G_T \) defined by \( G_T(\hat{a}_i) = a^T_i \) for \( i \in \{1, 2, 3\} \). Consider also the function \( \Theta_T : \hat{T} \rightarrow \mathbb{R}^2 \)

\[
\Theta_T(\hat{x}) := \frac{\hat{x}_1}{1-\hat{x}_2} \left( \varphi(\hat{x}_2) - (1-\hat{x}_2)a^2_T - \hat{x}_2a^3_T \right),
\]

where the limiting value has to be taken as \( \hat{x}_2 \) goes to 1. Then, there exists \( h_0 > 0 \) such that if \( h \in (0, h_0) \), \( T \) is the range of \( \hat{T} \) by the \( C^\infty \) and one-to-one mapping \( F_T : \hat{T} \rightarrow \mathbb{R}^2 \) given by

\[
F_T := G_T + \Theta_T.
\]

Moreover, each side of \( \hat{T} \) is mapped onto the corresponding side of \( T \), i.e., \( \Theta_T(0,t) = \Theta_T(t,0) = (0,0) \) and \( F_T(t,1-t) = \varphi(t) \), for all \( t \in [0,1] \). This type of diffeomorphism was first proposed by Zlamal [20] and studied by Scott [17]. If \( T \) is a straight (interior) triangle, we take the curving perturbation \( \Theta_T \equiv 0 \) and thus \( F_T \) is the usual affine map from the reference triangle, this hypothesis will be implicit in the following.

When \( T \) is a curved triangle, we need estimates on the derivatives of \( F_T \) and \( F_T^{-1} \) in order to obtain the usual scaling arguments. Such estimates are a consequence of

\[
|\Theta_T|_{m,\infty,\hat{T}} \leq C h_T^{\max(2,m)}, \quad (m \geq 1)
\]

which is proven in Theorem 22.4 of [19] (cf. also [17]) together with the following results.

**Lemma 5.** For all \( h \in (0, h_0) \), the Jacobian \( J_T \) of \( F_T \) does not vanish on a neighborhood of \( \hat{T} \) and the following estimates hold:

\[
C_1 h_T^2 \leq |J_T(\cdot)| \leq C_2 h_T^2, \quad (26)
\]

\[
|B_T|_{m,\infty,\hat{T}} \leq C h_T^{m+1}, \quad |B_T^{-1}|_{m,\infty,T} \leq C h_T^{-1}, \quad (m \geq 0),
\]

where \( B_T := DF_T \).
It follows from (26–27) and a careful application of the chain rule that (see [19] Lemma 25.1 and [1])

\[ \|u\|_{m,T} \leq C h_T^{2-m} \|u \circ F_T\|_{m,T}, \quad \|u \circ F_T\|_{m,T} \leq C h_T^{m-1} \|u\|_{m,T}, \]  

for all \( u \) in \( H^m(T) \) with \( m \geq 0 \).

Let us denote by \((\hat{T}; P_m(\hat{T}), \Sigma_m)\) the standard Lagrange finite element of order \( m \) on the reference triangle \( \hat{T} \). A finite element is defined on \( T \) by a triplet \((T, P_m(T), \Sigma_m)\), where \( P_m(T) \) is the image under \( F_T \) of the space \( P_m(\hat{T}) \) of polynomials of degree no greater than \( m \) on \( \hat{T} \):

\[ P_m(T) := \{ p : T \to \mathbb{C}; \ p = \hat{p} \circ F_T^{-1}, \ \hat{p} \in P_m(\hat{T}) \}, \]

and \( \Sigma_T = \{ N_i^k; \ i = 1, \cdots, (m+1)(m+2)/2 \} \) is a set of linear functionals defined by \( N_i(\phi) = \phi \circ F_T(\hat{a}_i), \ \forall \phi \in C^0(T) \) where \( \hat{a}_i \) are the nodes in \( \hat{T} \).

Interpolation error bounds on curved triangles are obtained by means of (28) and the technique used generally in the affine case, cf. [4] or [19]. Namely, if \( h \) is sufficiently small, one may readily prove with the aid of Lemma 5 that there exists a constant \( C \) independent of \( T \) such that

\[ \|v - \pi_T^m v\|_{1,T} \leq C h_T^m \|v\|_{m+1,T} \quad \forall v \in H^{m+1}(T), \quad (m \geq 1), \]  

where \( \pi_T^m v \in P_m(T) \) is uniquely determined by

\[ \pi_T^m v(\hat{a}_i^T) = v \circ F_T(\hat{a}_i), \quad \forall i = 1, \cdots, (m+1)(m+2)/2. \]

Notice that the norm \( \|\cdot\|_{m+1,T} \) in (29) may be substituted by the seminorm \( |\cdot|_{m+1,T} \) when \( T \) is a straight triangle.

We introduce the finite element spaces

\[ V_h^s := \{ v \in C^0(\Omega_h); \ v|_T \in P_1(T) \ \forall T \in \tau_h^s \} \]

and

\[ V_h^f := \{ q \in C^0(\Omega_h^-); \ q|_T \in P_1(T) \ \forall T \in \tau_h^f \}. \]

We deduce from (29) that

\[ \inf_{v \in (V_h^s)^2} \|u - v\|_{1,\Omega} \leq C h_T^m \|u\|_{m+1,\Omega}, \quad \forall u \in H^{m+1}(\Omega), \quad (m \geq 1) \]  

and

\[ \inf_{q \in V_h^f} \|p - q\|_{1,\Omega} \leq C h_T^m \|p\|_{m+1,\Omega}, \quad \forall p \in H^{m+1}(\Omega^-), \quad (m \geq 1). \]

Finally, for any integer \( n \), we consider the \( 2n \)-dimensional space

\[ T_n := \left\{ \sum_{j=0}^n a_j \cos jt + \sum_{j=1}^{n-1} b_j \sin jt; \ a_j, b_j \in \mathbb{C} \right\}. \]

We have the following approximation property (cf. [16]):

\[ \inf_{\mu \in T_n} \|\lambda - \mu\|_s \leq 2^{l-s} n^{s-t} \|\lambda\|_t \quad \forall \lambda \in H^l(0,2\pi) \quad \forall t \geq s. \]
The discrete problem

The discrete version of (11) is given by

\[
\begin{cases}
\text{find } u_h \in (V_h^s)^2, \ p_h \in V_h^f, \ \xi_n \in T_n \text{ and } \psi_n \in T_n \text{ such that} \\
E^w(u_h, v) + D(v, p_h) = L(v) \quad \forall v \in (V_h^s)^2 \\
a^k(p_h, q) + \rho_f \omega^2 D(u_h, q) - \int_0^{2\pi} \xi_n \gamma q dt = \ell(q) \quad \forall q \in V_h^f \\
\int_0^{2\pi} \mu \gamma p_h dt - \frac{1}{2} \int_0^{2\pi} \mu \psi_n dt - b(\psi_n, \mu) = 0 dt \quad \forall \mu \in T_n \\
\int_0^{2\pi} \xi_n \varphi dt + c(\psi_n, \varphi) = 0 \quad \forall \varphi \in T_n 
\end{cases}
\]

(33)

Let us denote \( \delta = \left( h, \frac{1}{n} \right) \) and \( M_\delta = (V_h^s)^2 \times V_h^f \times T_n \times T_n \). The discrete Inf-Sup condition

\[
\sup_{v \in M_\delta} \frac{|\langle A_0 u, v \rangle|}{\|v\|_M} \geq \frac{1}{C_0} \|u\|_M \quad \forall u \in M_\delta.
\]

(34)

is a consequence of the following Lemma.

**Lemma 6.** There exists a constant \( C_0 > 0 \) independent of \( \delta \) such that for all \( \hat{L} \in M' \) and for all \( \hat{u}_\delta \in M_\delta \) satisfying

\[
[A_0 \hat{u}, v] = [\hat{L}, v] \quad \forall v \in M_\delta
\]

(35)

we have

\[
\|\hat{u}_\delta\|_M \leq C_0 \|\hat{L}\|_{M'}.
\]

**Proof.** Let \( \Pi_n : L^2(0, 2\pi) \rightarrow T_n \) be the \( L^2(0, 2\pi) \)–orthogonal projection onto \( T_n \). This operator is characterized by

\[
\int_0^{2\pi} f(\Pi_n \varphi) dt = \int_0^{2\pi} f \varphi dt \quad \forall \varphi \in H^{1/2}, \ \forall f \in L^2(0, 2\pi).
\]

Furthermore, we deduce from (32) that

\[
\|\Pi_n \varphi\|_{1/2} \leq \|\varphi\|_{1/2} \quad \forall \varphi \in H^{1/2}.
\]

The two last properties of \( \Pi_n \) permit us to obtain the Inf-Sup condition

\[
\sup_{\varphi \in T_n} \frac{|\int_0^{2\pi} \xi \varphi dt|}{\|\varphi\|_{1/2}^{1/2}} \geq C_1 \|\xi\|_{-1/2} \quad \forall \xi \in T_n
\]

(36)

by using Fortin’s trick [2].
Let $\hat{u}_\delta = (u_h, p_h, \xi, \psi_n) \in M_\delta$ be a solution of (35). Identity

$$2(\pi - 1) \int_0^{2\pi} \gamma p_h \, dt + \int_0^{2\pi} \xi_n \, dt + \int_0^{2\pi} \psi_n \, dt = \ell(1) + 2 \int_0^{2\pi} g \, dt - 2 \int_0^{2\pi} f \, dt$$

is obtained as in the continuous case. Now, we introduce the variable $\phi_n = \psi_n - \Pi_n f$ and consider the subspace

$$M_{0,\delta} = (V^s_h)^2 \times V_{0,h}^f \times T_{0,n} \times T_{0,n}$$

where $V_{0,h}^f = \{ q \in V_h^f; \int_0^{2\pi} \gamma q \, dt = 0 \}$ and $T_{0,n} = T_n \cap H^1_0$. If we define the functions

$$p_{0,h}, \xi_{0,n} \text{ and } \phi_{0,h} \text{ by}$$

$$p_{0,h} = p_h - \frac{1}{2\pi} \int_0^{2\pi} \gamma p_h \, dt, \quad \xi_{0,n} = \xi_n - \frac{1}{2\pi} \int_0^{2\pi} \xi_n \, dt \quad \text{and} \quad \phi_{0,h} = \phi_n - \frac{1}{2\pi} \int_0^{2\pi} \phi_n \, dt,$$

then, it is straightforward that $\hat{u}_{0,\delta} : = (u_h, p_{0,h}, \xi_{0,n}, \phi_{0,h}) \in M_{0,\delta}$ satisfies

$$[A_{0}\hat{u}_{0,\delta}, \hat{v}] = [\hat{L}, \hat{v}] \quad \forall \hat{v} \in M_{0,\delta},$$

(37)

where $\hat{L} = (L, \ell, 0, \tilde{g})$ with $\tilde{g} = g - \mathcal{H}_0(\Pi_n f)$. Following verbatim the steps given in Theorem 3 and using (36) we obtain the result.

Theorem 7. Assume that $-k^2$ is not an eigenvalue of the Laplacian in $\Omega$ with a Neumann boundary condition on $\Gamma$ and that problem (3) admits only the trivial solution. Then, for all $\delta$ small enough, problem (33) has a unique solution $\hat{u}_\delta$. Moreover, the Galerkin method is stable and we have Céa's estimate

$$\|\hat{u} - \hat{u}_\delta\|_{M} \leq C_1 \inf_{\hat{v} \in M_{\delta}} \|\hat{u} - \hat{v}\|_{M},$$

where $\hat{u} = (u, p, \xi, \psi)$ is the solution of (11).

In case $u \in H^{m+1}(\Omega)^2$ and $p \in H^{m+1}(\Omega^\gamma)$ we have

$$\|\hat{u} - \hat{u}_\delta\|_{M} \leq C_2 \left(h^m(\|u\|_{m+1,\Omega_s} + \|p\|_{m+1,\Omega_s}) + (2/n)^\sigma(\|\xi\|_{\sigma-1/2} + \|\psi\|_{\sigma+1/2})\right)$$

for all $\sigma > 0$.

Proof. The theorem is a consequence of a classical result for compact perturbations of operator equations. Indeed, $A_0$ satisfies the continuous and the discrete Inf-Sup conditions (17) and (34). Furthermore, the approximation properties (31) and (32) and the density of smooth functions in $M$ yield

$$\lim_{\delta \to 0} \inf_{v \in M_\delta} \|z - v\|_{M} = 0.$$  

(38)

On the other hand we also proved that problem (11) is also well posed and that it is a compact perturbation of $A_0$ (see Theorem 4). Under these hypotheses, Theorem 13.7 of [9] shows that, if $\delta$ is sufficiently small, (33) is also well posed and convergent. Finally, the convergence implies Céa’s estimate and the last assertion of the theorem follows from the approximation properties (31) and (32).
8 Description of the fully discrete method

8.1 Approximation of $a^k(\cdot, \cdot)$ on $V_h \times V_h$

Consider first a quadrature formula on the reference triangle

$$\hat{Q}(\phi) := \sum_{l=1}^{L} \hat{\omega}_l \phi(\hat{z}_l) \simeq \int_{\hat{T}} \phi$$

that is exact on $P_{2n-2}(\hat{T})$ and with weights $\hat{\omega}_l > 0$. On each $T \in \hat{\tau}_h^f$ we define

$$Q_T(\phi) := \hat{Q}(|J_T|\phi \circ F_T) = \sum_{l=1}^{L} \hat{\omega}_l |J_T(\hat{z}_l)|\phi(F_T(\hat{z}_l)) \simeq \int_T \phi(x) \, dx.$$

This induces us to define an approximation $a^k_h(\cdot, \cdot)$ of $a^k(\cdot, \cdot)$ by:

$$a^k_h(p, q) = \sum_{T \in \hat{\tau}_h^f} Q_T(\nabla p \cdot \nabla q - k^2 \theta pq) \quad \forall p, q \in V^f_h.$$

8.2 Approximation of $b(\cdot, \cdot)$ on $T_n \times T_n$

For all continuous and $2\pi$–periodic function $g$ we consider the composite trapezoidal rule

$$Q_n(g) := \frac{\pi}{n} \sum_{i=0}^{2n-1} g\left(\frac{i\pi}{n}\right)$$

associated to the partition of $[0, 2\pi]$ into $2n$ grid points.

We also need to construct approximations for the improper integral

$$(\Lambda_0 g)(t) := -\frac{1}{2\pi} \int_{0}^{2\pi} \log\left(\frac{4}{e} \sin^2 \frac{t-s}{2}\right) g(s) \, ds. \quad (39)$$

We can proceed as in [9] and obtain a quadrature formula replacing $g(s)$ in (39) by its trigonometric interpolation polynomial

$$(P_n g)(s) := \sum_{j=0}^{2n-1} g\left(\frac{j\pi}{n}\right)L_j(s)$$

where the Lagrange basis is given by

$$L_j(s) := \frac{1}{2n} \left\{ 1 + 2 \sum_{k=1}^{n-1} \cos k(s - \frac{j\pi}{n}) + \cos n(s - \frac{j\pi}{n}) \right\}, \quad (j = 0, \ldots, 2n - 1).$$
Therefore, we obtain the formula
\[
\tilde{Q}_n g(t) := \sum_{j=0}^{2n-1} R_j^{(n)}(t) g(\frac{j\pi}{n})
\]
where, for \(j = 0, \cdots, 2n - 1\), the weights
\[
R_j^{(n)}(t) = \frac{1}{2n} + \frac{1}{n} \sum_{m=1}^{n-1} \frac{1}{m} \cos m(t - \frac{j\pi}{n}) + \frac{1}{2n^2} \cos n(t - \frac{j\pi}{n})
\]
are given explicitly by evaluating the integrals \((\Lambda_0 L_j)(t)\) with the aid of (14), cf. [9]. Using the splitting (cf. [5])
\[
K(t, s) = -\frac{1}{2\pi} K_1(t, s) \log(\frac{4}{e} \sin^2 \frac{t-s}{2}) + K_2(t, s), \tag{40}
\]
of the double layer acoustic potential, where
\[
K_1(s, t) := -\frac{k}{2} J_1(k|\mathbf{x}(t) - \mathbf{x}(s)|) \frac{x_2'(s)(x_1(t) - x_1(s)) - x_1'(s)(x_2(t) - x_2(s))}{|\mathbf{x}(t) - \mathbf{x}(s)|},
\]
and \(J_1\) is the Bessel function of order one, we obtain
\[
b(\xi, \mu) = \int_0^{2\pi} \Lambda_0(K_1(t, \cdot)\xi(\cdot))(t) \mu(t) \, dt + \int_0^{2\pi} \left( \int_0^{2\pi} K_2(t, s)\xi(s) \, ds \right) \mu(t) \, dt. \tag{41}
\]
Hereafter, taking into account that \(K_1\) and \(K_2\) are in \(C_2^{\infty}\) with respect to each variable, the first term of the right hand side in (41) may be approximated by using the quadrature rule \(\tilde{Q}_n\) for the internal integral and \(Q_n\) for the external one. The two-dimensional quadrature rule derived from \(Q_n\) is applied to the second term. In other words, we are introducing an approximation of the bilinear form \(b(\cdot, \cdot)\) on \(T_n \times T_n\) given by
\[
b_n(\xi, \mu) := Q_n[\tilde{Q}_n[K_1(t, \cdot)\xi(\cdot)]\mu(t)] + Q_n[Q_n[K_2(t, \cdot)\xi(\cdot)]\mu(t)]
\]
which may also be written in matricial form as follows:
\[
b_n(\xi, \mu) = \sum_{i=0}^{2n-1} \left( \sum_{j=0}^{2n-1} B_{i,j} \xi(\frac{j\pi}{n}) \right) \mu(\frac{i\pi}{n})
\]
The entries of the \(2n \times 2n\) matrix \(B\) are
\[
B_{i,j} := \frac{\pi}{n} R_j^{(n)}(i\pi\frac{\pi}{n}) K_1(i\pi\frac{\pi}{n}, j\pi\frac{\pi}{n}) + \frac{\pi^2}{n^2} K_2(i\pi\frac{\pi}{n}, j\pi\frac{\pi}{n}).
8.3 Approximation of $c(\cdot, \cdot)$ on $T_n \times T_n$

We deduce from identity (8) that

$$c(\xi, \mu) = c'(\xi, \mu) - k^2 \bar{c}(\xi, \mu)$$

where

$$c'(\xi, \mu) = \int_0^{2\pi} \left( \int_0^{2\pi} V(t, s) \xi'(s) \, ds \right) \mu'(t) \, dt$$

and

$$\bar{c}(\xi, \mu) = \int_0^{2\pi} \left( \int_0^{2\pi} V(t, s) x' \cdot x'(s) \, ds \right) \mu(t) \, dt$$

For both bilinear forms we use the splitting (cf. [5])

$$V(t, s) = -\frac{1}{2\pi} V_1(t, s) \log \left( \frac{4}{e} \sin^2 \frac{t-s}{2} \right) + V_2(t, s), \quad (42)$$

of the single layer acoustic potential, where $V_1(t, s) := \frac{1}{2} J_0(k|x(t) - x(s)|)$ and $J_0$ is the Bessel function of order zero. Let $C$ be the $2n \times 2n$ matrix

$$C_{i,j} := \frac{\pi}{n} R_j^{(n)}(i\pi \frac{n}{n}) V_1(i\pi \frac{n}{n}, j\pi \frac{n}{n}) + \frac{\pi^2}{n^2} V_2(i\pi \frac{n}{n}, j\pi \frac{n}{n}) \quad (0 \leq i, j \leq 2n - 1).$$

We begin by seeking and approximation of $\bar{c}(\cdot, \cdot)$. We proceed as for $b(\cdot, \cdot)$ and define

$$\bar{c}_n(\xi, \mu) = \sum_{i=0}^{2n-1} \left( \sum_{j=0}^{2n-1} \bar{C}_{i,j} \xi(i\pi \frac{n}{n}) \mu(j\pi \frac{n}{n}) \right)$$

where

$$\bar{C}_{i,j} := x'(i\pi \frac{n}{n}) \cdot x'(j\pi \frac{n}{n}) C_{i,j} \quad (0 \leq i, j \leq 2n - 1).$$

Similarly, we take

$$c'_n(\xi, \mu) = \sum_{i=0}^{2n-1} \left( \sum_{j=0}^{2n-1} C_{i,j} \xi'(j\pi \frac{n}{n}) \mu'(i\pi \frac{n}{n}) \right).$$

Now, to express the derivatives $\mu'(i\pi \frac{n}{n})$ in terms of $\{\mu(j\pi \frac{n}{n}), j = 1, \cdots, 2n - 1\}$ we use the matrix $D = (L'_j(i\pi \frac{n}{n}))_{0 \leq i, j \leq 2n - 1}$. Straightforward computations show that $D_{i,j} = 0$ and

$$D_{i,j} = \frac{(-1)^{|i-j|}}{2} \cot \left( \frac{(i-j)\pi}{2n} \right) \quad (0 \leq i \neq j \leq 2n - 1).$$

Finally, we arrive to the following approximation of $c(\cdot, \cdot)$

$$c_n(\xi, \mu) = \sum_{i=0}^{2n-1} \left( \sum_{j=0}^{2n-1} H_{i,j} \xi(j\pi \frac{n}{n}) \mu(i\pi \frac{n}{n}) \right)$$

where $H = -DCD + \tilde{C}$. 
8.4 Boundary integrals computed explicitly

We still have to compute two boundary bilinear forms. the first one is given explicitly by

$$\int_0^{2\pi} \xi \mu \, dt = \sum_{i=0}^{2n-1} \left( \sum_{j=0}^{2n-1} M_{i,j} \xi \left( \frac{j\pi}{n} \right) \right) \mu \left( \frac{i\pi}{n} \right) \quad (\mu, \xi \in T_n),$$

where

$$M = \left( \int_0^{2\pi} L_j(t) L_m(t) \, dt \right)_{0 \leq j,m \leq 2n-1} = \left( \frac{\pi}{n} \delta_{jm} - \frac{(-1)^{j-m}}{2n} \right)_{0 \leq j,m \leq 2n-1}.$$

For the last boundary bilinear form we first notice that the parameterized trace $\gamma q$ of any $q \in V_h^f$ belongs to the space $T_h^m$ of $2\pi$-periodic and continuous functions whose restriction to any $(\frac{m\pi}{N}, \frac{(m+1)\pi}{N})$ is polynomial of degree $\leq m$:

$$T_h^m = \left\{ \phi \in C^0_{2\pi}; \quad \phi \mid_{(\frac{m\pi}{N}, \frac{(m+1)\pi}{N})} \in P_m \quad \forall i = 0, \ldots, 2N-1 \right\}.$$

Let us denote by $\{ \ell_p, \quad p = 1, \ldots, \dim(T_h^m) \}$ the usual nodal basis of $T_h^m$. Then

$$\int_0^{2\pi} \mu \gamma q \, dt = \sum_{i=0}^{2n-1} \left( \sum_{j=0}^{2n-1} R_{i,j} \gamma \left( \frac{j\pi}{N} \right) \right) \mu \left( \frac{i\pi}{n} \right) \quad (\mu \in T_n, \quad q \in V_h^f),$$

where

$$R_{i,j} = \int_0^{2\pi} \ell_j(t) L_i(t) \, dt \quad 0 \leq i \leq 2n-1, \quad 0 \leq j \leq 2N-1.$$

Notice that there exists an index $p \in \{0, \ldots, 2N-1 \}$ such that the support of $\ell_j$ is contained in $(\frac{m\pi}{N}, \frac{(m+1)\pi}{N})$. Thus the entries of $R$ may be obtained explicitly by performing $m$ integrations by parts in each term of the decomposition

$$\int_0^{2\pi} \ell_j(t) L_i(t) \, dt = \int_{\frac{m\pi}{N}}^{\frac{(m+1)\pi}{N}} \ell_j(t) L_i(t) \, dt + \int_{\frac{m\pi}{N}}^{\frac{(m+1)\pi}{N}} \ell_j(t) L_i(t) \, dt.$$

We are now in a position to propose a completely discrete versions of the Galerkin scheme (33):

\[
\begin{align*}
\text{find } u_h^s \in (V_h^s)^2, \ p_h^s \in V_h^f, \ \xi_h^s \in T_n \text{ and } \psi_h^s \in T_n \text{ such that } \\
E^\varphi(u_h^s, v) + D(v, p_h^s) = L(v) \quad \forall v \in (V_h^s)^2 \\
\alpha_h^k(p_h^s, q) + \rho_f \omega^2 D(u_h^s, q) - \int_0^{2\pi} \xi_h^s q \, dt = \ell(q) \quad \forall q \in V_h^f \\
\int_0^{2\pi} \mu \gamma p_h^s \, dt - \frac{1}{2} \int_0^{2\pi} \mu \psi_h^s \, dt - b_n(\psi_h^s, \mu) = 0 \quad \forall \mu \in T_n \\
\int_0^{2\pi} \xi_h^s \varphi \, dt + c_n(\psi_h^s, \varphi) = 0 \quad \forall \varphi \in T_n.
\end{align*}
\]
8.5 Matrix form of the fully discrete problem

Let us denote by \( \{ \varphi_s^i, i = 1, \ldots, M_s^h \} \) and \( \{ \varphi_f^i, i = 1, \cdots, M_f^h \} \) the nodal basis of \( V_s^h \) and \( V_f^h \) respectively. We also consider the canonical basis \( \{ \mathbf{e}_1 := (1, 0), \mathbf{e}_2 := (0, 1) \} \) of \( \mathbb{R}^2 \).

For \( 1 \leq \alpha, \beta \leq 2 \) we denote by \( E^{\alpha\beta}_{ij} := E(\varphi_s^i \mathbf{e}_\alpha, \varphi_s^j \mathbf{e}_\beta) \).

Let us also introduce the \( M_s^h \times M_f^h \) matrix \( D^\alpha (\alpha = 1, 2) \)

\[
D_{ij}^\alpha := D(\varphi_s^i \mathbf{e}_\alpha, \varphi_f^j).
\]

If we set

\[
\begin{aligned}
\mathbf{u}_h^* &= \sum_{\alpha=1}^{2} \sum_{i=1}^{M_s^h} \mathbf{u}_i^{(\alpha)} \varphi_s^i \mathbf{e}_\alpha,
\mathbf{p}_h^* &= \sum_{i=1}^{M_f^h} \mathbf{p}_i \varphi_f^i,
\mathbf{\xi}_n^* &= \sum_{i=0}^{2n-1} \mathbf{\xi}_i \mathbf{L}_i,
\mathbf{\psi}_n(t) &= \sum_{i=0}^{2n-1} \mathbf{\psi}_i \mathbf{L}_i(t)
\end{aligned}
\]

and use the superscript \((\cdot)^T\) to denote transposition of matrices, then, the matricial interpretation of \((43)\) takes the form

\[
\begin{pmatrix}
E & D & 0 & 0 \\
\rho_j \omega^2 D^T & A - R^T & 0 \\
0 & R^T & 0 & -\frac{1}{2} M - B \\
0 & 0 & M & H
\end{pmatrix}
\begin{pmatrix}
\mathbf{u} \\
\mathbf{p} \\
\mathbf{\xi} \\
\mathbf{\psi}
\end{pmatrix}
= 
\begin{pmatrix}
\mathbf{F} \\
\mathbf{G}
\end{pmatrix}
\]

(44)

where

\[
E := \begin{pmatrix}
E^{11} & E^{12} \\
E^{21} & E^{22}
\end{pmatrix},
D := \begin{pmatrix}
D^{11} \\
D^{22}
\end{pmatrix}
\]

and

\[
A_{ij} := a_h^\alpha(\varphi_s^i, \varphi_f^j).
\]

The righthand side of \((44)\) is given by

\[
\begin{aligned}
\mathbf{F} := \begin{pmatrix}
\mathbf{F}_1^1 \\
\mathbf{F}_1^2
\end{pmatrix}
\text{ with } \mathbf{F}_1^\alpha := \mathbf{L}(\varphi_s^i \mathbf{e}_\alpha), \quad (\alpha = 1, 2), \quad (i = 1, \cdots, M_s^h),
\end{aligned}
\]

and

\[
\begin{aligned}
\mathbf{G}_i := \ell(\varphi_f^i), \quad (i = 1, \cdots, M_f^h).
\end{aligned}
\]

The matrix in \((44)\) is badly structured since \( A, D \) are sparse matrices while \( C \) and \( M \) are full. The global matrix is too large to be stored and handled. In view the convergence estimates obtained in Theorem 7, we expect that the number of degrees of freedom may drastically reduced without affecting the accuracy of the method, (see the numerical results presented in [13, 15]). This leads us to introduce the following strategy to solve...
the linear systems of equations. We eliminate the boundary variable from (44) to obtain the reduced system

\[
\begin{pmatrix}
E D \\
\rho_f \omega^2 D^T
\end{pmatrix}
\begin{pmatrix}
A + R'M^{-1}H(\frac{1}{2}M + B)^{-1}R \\
\end{pmatrix}
\begin{pmatrix}
u \\
p
\end{pmatrix}
= \begin{pmatrix}
F \\
\tilde{G}
\end{pmatrix}.
\] (45)

The system of equations (45) can be solved by the GMRES method preconditioned with the block diagonal matrix

\[
\begin{pmatrix}
E_0 & 0 \\
0 & A_0
\end{pmatrix},
\]

where \(E_0\) and \(A_0\) are the matrices associated to the sesquilinear forms \(\int_{\Omega} \sigma(u): \varepsilon(v) \, dx\) and \(\int_{\Omega} \nabla p \cdot \nabla q \, dx\) respectively.

Each iteration of the GMRES method entails the solution of two linear systems with full but small matrix \(M\) and \(\frac{1}{2}M + B\) and the solution of two other linear systems with sparse matrices \(A_0\) and \(R_0\). This can be performed by any of the numerous strategies existing in the literature for these standard stiffness matrices.

9 Analysis of the fully discrete method

We begin our analysis with the following classical result, see [4] or [19].

**Lemma 8.** There exists \(h_0 \in (0, 1]\) such that

\[
|a^k(p, q) - a_n^k(p, q)| \leq C h^m \|p\|_{1, \Omega^-} \|q\|_{1, \Omega^-} \quad \forall p, q \in V^f_h
\]

for some constant \(C > 0\) independent of \(h\), \(\forall h \leq h_0\).

**Lemma 9.** There exists a constant \(C\) independent of \(n\) such that for any \(\sigma > 0\)

\[
|b(\psi, \mu) - b_n(\psi, \mu)| \leq C n^{-\sigma} \|\psi\|_{1/2} \|\mu\|_{-1/2} \quad \forall \psi, \mu \in T_n
\]

and

\[
|c(\psi, \varphi) - c_n(\psi, \varphi)| \leq C n^{-\sigma} \|\psi\|_{1/2} \|\varphi\|_{1/2} \quad \forall \psi, \varphi \in T_n.
\]

**Proof.** The first estimate is a direct consequence of Lemma 7.3. of [15]

\[
|b(\psi, \mu) - b_n(\psi, \mu)| \leq C_1 n^{-\sigma} \|\psi\|_{-1/2} \|\mu\|_{-1/2} \leq C_1 n^{-\sigma} \|\psi\|_{1/2} \|\mu\|_{-1/2} \quad \forall \psi, \mu \in T_n,
\]

where the last inequality is a consequence of the imbedding \(H^{1/2} \hookrightarrow H^{-1/2}\).

On the other hand

\[
|c(\psi, \varphi) - c_n(\psi, \varphi)| \leq |c'(\psi, \varphi) - c'_n(\psi, \varphi)| + k^2 |\tilde{c}(\psi, \varphi) - \tilde{c}_n(\psi, \varphi)|
\]
and using again Lemma 7.3. of [15] for both terms we obtain
\[ |c'(\psi, \varphi) - c'_n(\psi, \varphi)| \leq C_2 n^{-\sigma} \|\psi\|_{-1/2} \|\varphi\|_{-1/2} \]
and
\[ |\bar{c}(\psi, \varphi) - \bar{c}_n(\psi, \varphi)| \leq C_3 n^{-\sigma} \|x'\psi\|_{-1/2} \|x'\varphi\|_{-1/2} \]
for all \(\psi, \varphi \in T_n\). Noticing that the derivative is continuous from \(H^{1/2}\) onto \(H^{-1/2}\) and that \(g \mapsto x'g\) maps continuously \(H^{-1/2}\) onto \((H^{-1/2})^2\) (see Lemma 5.13.1 of [16]) we deduce the second estimate of the Lemma.

We conclude our analysis by the following result.

**Theorem 10.** Assume that \(-k^2\) is not an eigenvalue of the Laplacian in \(\Omega\) with a Neumann boundary condition on \(\Gamma\) and that problem (3) admits only the trivial solution. Then, for \(\delta = (h, 1/n)\) small enough, the fully discrete scheme (43) is well-posed and convergent. Moreover, in case \(u \in H^{m+1}(\Omega_f)\) and \(p \in H^{m+1}(\Omega_f)\), for all \(\sigma > 0\) we have
\[
\|\tilde{u} - \tilde{u}_h\|_M \leq C \left( h^m \|u\|_{m+1, \Omega_f} + \|p\|_{m+1, \Omega_f} \right) + n^{-\sigma} (\|\xi\|_{\sigma-1/2} + \|\psi\|_{\sigma+1/2}).
\]

**Proof.** Let us denote
\[ A(\tilde{u}, \tilde{v}) = [A\tilde{u}, \tilde{v}] \quad \forall \tilde{u}, \tilde{v} \in M. \]
We introduce now the approximation \(A_\delta(\tilde{u}, \tilde{v})\) of \(A(\tilde{u}, \tilde{v})\) on \(M_\delta\) obtained by substituting \(a^k(\cdot, \cdot), b(\cdot, \cdot)\) and \(c(\cdot, \cdot)\) by \(a^h_k(\cdot, \cdot), b_n(\cdot, \cdot)\) and \(c_n(\cdot, \cdot)\) respectively.

On the one hand, the convergence and stability of the Galerkin method (33) (see Theorem 7) is equivalent to the uniform inf–sup condition
\[
\sup_{\tilde{v} \in M_\delta} \frac{A(\tilde{u}, \tilde{v})}{\|\tilde{v}\|_M} \geq C_0 \|\tilde{u}\|_M, \quad \forall \tilde{u} \in M_\delta.
\]
On the other hand, Lemmas 9 and 8 yield
\[
|A(\tilde{u}, \tilde{v}) - A_\delta(\tilde{u}, \tilde{v})| \leq C_1 (h^m + n^{-\sigma}) \|\tilde{u}\|_M \|\tilde{v}\|_M \quad \forall \tilde{u}, \tilde{v} \in M_\delta \tag{46}
\]
which permits us to deduce by standard arguments that, for \(\delta\) small enough, \(A_\delta(\cdot, \cdot)\) also satisfies a uniform inf–sup condition and hence problem (43) has a unique solution.

Now, the second Strang Lemma [18] gives the abstract estimate
\[
\|\tilde{u}_h^* - \tilde{v}\|_M \leq C_2 \left( \sup_{\tilde{z} \in M_\delta} \frac{|A(\tilde{u} - \tilde{z}, \tilde{v})|}{\|\tilde{z}\|_M} + \sup_{\tilde{z} \in M_\delta} \frac{|A(\tilde{v}, \tilde{z}) - A_\delta(\tilde{v}, \tilde{z})|}{\|\tilde{z}\|_M} \right),
\]
for all \(\tilde{v} \in M_\delta\).

We deduce from the triangle inequality
\[
\|\tilde{u} - \tilde{u}_h^*\|_M \leq \|\tilde{u} - \tilde{u}_h\|_M + \|\tilde{u}_h^* - \tilde{u}_h\|_M
\]
and the second Strang inequality, that the asymptotic behavior of the fully discrete method is a direct consequence of (46), the boundness of \(A(\cdot, \cdot)\) and Theorem 7. \(\square\)
A BEM–FEM METHOD FOR FLUID–SOLID INTERACTION PROBLEMS

References


